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PERIODIC STREAM LINES IN THE THREE-DIMENSIONAL SQUARE CELL PATTERN *

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Abstract: *By using the theory of the generalized perturbed Hamiltonian systems, it is shown that there exist periodic stream lines in the three-dimensional square cell pattern of Rayleigh-Benard convection. The result means that our method enables this three-dimensional flow pattern to be described in an unambiguous manner, and some experimental results of other authors can be explained.*

Key words: Rayleigh-Benard convection; periodic solutions; Hamiltonian systems; square cell pattern

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Introduction

In the theory of thermal convection instability between two horizontal plane, many planform studies of Rayleigh-Benard convection have been published over the past 30 years. D. R. Jerkins[1] said: "On the theoretical side, interest has been in the competition between the roll, square and hexagonal planforms for steady convection. On the experimental side, convection planforms are often observed using a shadowgraph, which is a two-dimensional image of the planform derived from a three-dimensional field." The paper [1] shows that the shadowgraph technique does produce an unambiguous pattern for the square solution, as indeed it does for the hexagonal and roll solutions, and the cell boundaries are in fact the surfaces on which fluid is moving downwards. These conclusions tell us that the assertion given by Stuart[2] that the square cell pattern is not a valid mathematical model that could be observed is incorrect. So that it is necessary to investigate continuously the three-dimensional square cell pattern. Since three-dimensional dynamical system introduces us to a qualitatively new phenomena, the existence of stream lines chaotically arranged in space, which is sometimes called Lagrangian turbulence, the study of periodic and chaotic phenomena plays an important role in the understanding of the onset of turbulence. What are the structural properties of steady-state three-dimensional flows and how

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do we check the creation of the periodic stream-lines? The purpose of this paper is to answer the existence of periodic solutions for the three-dimensional square model. To our knowledge, no pure mathematical investigation on this model has been published.

In order to study the above model, we now outline the Melnikov analytical technique developed by [3] for the generalized perturbed Hamiltonian systems.

A three-dimensional perturbed generalized Hamiltonian system is taken to be of the following form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \{x_i, H\}(\boldsymbol{x}) + \varepsilon g_i(\boldsymbol{x}, t), \qquad i = 1,2,3, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, $\{\cdot, \cdot\}$ denotes the Poisson bracket and $H \in C^r(\mathbf{R}^3)$, $r \geqslant 3$, is called the Hamiltonian function. We assume that each g_i is sufficiently smooth ($C^r, r \geqslant 2$). Let

$$f_i(x) = \{x_i, H\}(\boldsymbol{x}) = \sum_{j=1}^{3} J_{ij}(\boldsymbol{x}) \frac{\partial H}{\partial x_j}(\boldsymbol{x}), \qquad i = 1,2,3,$$

where $J_{ij}(\boldsymbol{x}) = \{x_i, x_j\}$. The 3×3 anti-symmetric matrix $\boldsymbol{J}(\boldsymbol{x}) = (J_{ij}(\boldsymbol{x}))$ is called the structure matrix of a generalized Hamiltonian system. We note that the system $(1)_\varepsilon$ can be rewritten in the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{J}(\boldsymbol{x}) \nabla H(\boldsymbol{x}) + \varepsilon \boldsymbol{g}(\boldsymbol{x}, t),$$

where $\boldsymbol{g} = (g_1, g_2, g_3)$. Suppose that the perturbation terms $g_i(x, t)$, $i = 1,2,3$, are T-periodic with respect to t, and $0 \leqslant \varepsilon \leqslant 1$.

We make the following assumptions about the unperturbed system $(1)_{\varepsilon=0}$:

(A_1) We assume that there exists a Casmir function C of the Poisson manifold ($\mathbf{R}^3$, $\{\cdot, \cdot\}$), which is defined on the regular point set $M = \{\boldsymbol{x} \in \mathbf{R}^3 : \mathrm{rank}\boldsymbol{J}(\boldsymbol{x}) = 2\}$ (or a connected open subset $U \subset M$), such that the gradient $\nabla C(\boldsymbol{x}) \neq 0$ for each $\boldsymbol{x} \in M_c = \{\boldsymbol{x} \in M : C(\boldsymbol{x}) = c\}$ with constant c satisfying $|c| \leqslant \delta$ for some $\delta > 0$.

(A_2) In addition to assumption (A_1), we assume that for each value of c in some open interval $I \subset \{c \in \mathbf{R} : |c| \leqslant \delta\}$, on the level set M_c the unperturbed system $(1)_\varepsilon$ possesses a one parameter family of periodic orbits, $q_0^\alpha(t - \theta, c)$, $\alpha \in L(c)$, where $L(c) \subset \mathbf{R}$ is an open interval and θ denotes the "phase" or starting point of the orbit. We denote the period of $q_0^\alpha(t - \theta, c)$ by $T(\alpha, c)$.

Under the above assumptions, we study the existence of periodic orbits of the perturbed system $(1)_{\varepsilon=0}$. We want to determine if the two parameter family of periodic orbits in the unperturbed system persist as periodic orbits in the perturbed system. To do this, we rewrite (1) as an autonomous system of differential equations, where we denote $\boldsymbol{f} = (f_1, f_2, f_3)$ and $\boldsymbol{g} = (g_1, g_2, g_3)$. We define the function $\Phi(t) = t(\mathrm{mod}\ T)$; by the T-periodicity of g we then have the following suspended system:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) + \varepsilon \boldsymbol{g}(\boldsymbol{x}, \Phi), \tag{2}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 1, \qquad (\boldsymbol{x}, \Phi) \in \mathbf{R}^3 \times S_T,$$

where $S_T = R(\mathrm{mod}\ T)$. Further we shall reduce our study to a three-dimensional Poincaré map. Define a global cross-section transverse to the vector field of $(2)_\varepsilon$ by

$$\Sigma^0 = \{(\boldsymbol{x}, \Phi) \in \mathbf{R}^3 \times S_T : \Phi = 0(\mathrm{mod}\ T)\},$$

and define the mth Poincaré map:

$$P_\varepsilon^m : \Sigma^0 \to \Sigma^0,$$

by

$$P_\varepsilon^m : x_\varepsilon(0) \to x_\varepsilon(mT).$$

Thus, the study of subharmonic periodic orbits of $(2)_\varepsilon$ is reduced to the study of fixed points of the map P_ε^m.

By a transformation of the form

$$\boldsymbol{x} = (x_1, x_2, x_3) \to (a(\boldsymbol{x}), \phi(\boldsymbol{x})(\mathrm{mod}\ 2\pi), C(\boldsymbol{x})),$$

the system $(2)_\varepsilon$ becomes the following system:

$$\left.\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \varepsilon\langle \nabla a, g\rangle(a, \phi, c, \Phi) \equiv \varepsilon F(a, \phi, c, \Phi),\\
\frac{\mathrm{d}\phi}{\mathrm{d}t} &= \left.\frac{\partial \hat{H}}{\partial a}\right|_c + \varepsilon\langle \nabla \phi, g\rangle(a, \phi, c, \Phi) \equiv \Omega(a, c) + \varepsilon G(a, \phi, c, \Phi),\\
\frac{\mathrm{d}c}{\mathrm{d}t} &= \varepsilon\langle \nabla C, g\rangle(a, \phi, c, \Phi) \equiv \varepsilon R(a, \phi, c, \Phi),\\
\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= 1,
\end{aligned}\right\} \tag{3}$$

where $(a, \phi, c, \Phi) \in A \times S_{2\pi} \times I \times S_T$.

We can easily solve the unperturbed system $(3)_{\varepsilon=0}$ (without loss of generality, take the initial value $\Phi(0) = 0$) and obtain the solution

$$a = 0, \quad \phi = \Omega(a_0, c_0)t + \phi_0(\mathrm{mod}\ 2\pi),$$

$$c = c_0, \quad \Phi = t(\mathrm{mod}\ 2\pi).$$

Thus, the coordinate a plays the role of the parameter α in assumption (A_2).

Now we construct an approximation to the Poincaré map. The cross-section to the flow defined by $(3)_\varepsilon$ is

$$\hat{\Sigma}^0 = \{(a, \phi, c, \Phi) \in A \times S_{2\pi} \times I \times S_T : \Phi = 0(\mathrm{mod}\ T)\},$$

and the mth iterate of the Poincaré map, $\hat{P}_\varepsilon^m$, in the new coordinates is

$$\hat{P}_\varepsilon^m : (a_\varepsilon(0), \phi_\varepsilon(0), c_\varepsilon(0)) \to (a_\varepsilon(mT), \phi_\varepsilon(mT), c_\varepsilon(T)).$$

where $(a_\varepsilon(t),\ \phi_\varepsilon(t),\ c_\varepsilon(t),\ \Phi(t))$ is the solution of the system $(3)_\varepsilon$ starting from $(a_0, \phi_0, c_0, 0) \in \hat{\Sigma}^0$. Using the solution to the unperturbed system, we can approximate the Poincaré map using regular perturbation theory

$$a_\varepsilon(t) = a_0 + \varepsilon a_1(t) + O(\varepsilon^2),$$

$$\phi_\varepsilon(t) = \Omega(a_0, c_0)t + \phi_0 + \varepsilon\phi_1(t) + O(\varepsilon^2),$$

$$c_\varepsilon(t) = c_0 + \varepsilon c_1(t) + O(\varepsilon^2),$$

where $t \in [0, \hat{T}(a_0, c_0)]$ in which $\hat{T}(a_0, c_0) = (2\pi)/\Omega(a_0, c_0)$, and a_1, ϕ_1 and c_1 satisfy the variational system

$$\frac{da_1}{dt} = F(a_0, \Omega(a_0, c_0)t + \phi_0, c_0, t),$$

$$\frac{d\phi_1}{dt} = \frac{\partial \Omega}{\partial a}(a_0, c_0)a_1(t) + \frac{\partial \Omega}{\partial c}(a_0, c_0)c_1(t) + G(a_0, \Omega(a_0, c_0)t + \phi_0, c_0, t),$$

$$\frac{dc_1}{dt} = R(a_0, \Omega(a_0, c_0)t + \phi_0, c_0, t),$$

and hence (choosing $a_1(0) = \phi_1(0) = c_1(0) = 0$)

$$a_1(mT) = \int_0^{mT} F(a_0, \Omega(a_0, c_0)t + \phi_0, c_0, t)dt \equiv \overline{M}_1^{m/n}(a_0, \phi_0, c_0),$$

$$\phi_1(mT) = \frac{\partial \Omega}{\partial a}(a_0, c_0)\int_0^{mT}\int_0^{t} F(a_0, \Omega(a_0, c_0)\eta + \phi_0, c_0, \eta)d\eta dt + \frac{\partial \Omega}{\partial c}(a_0, c_0)\int_0^{mT}\int_0^{t} R(a_0, \Omega(a_0, c_0) + \phi_0, c_0, \eta)d\eta dt + \int_0^{mT} G(a_0, \Omega(a_0, c_0)t + \phi_0, c_0, t)dt \equiv \overline{M}_2^{m/n}(a_0, \phi_0, c_0),$$

$$c_1(mT) = \int_0^{mT} R(a_0, \Omega(a_0, c_0)t + \phi_0, c_0, t)dt \equiv \overline{M}_3^{m/n}(a_0, \phi_0, c_0).$$

We define the vector $\overline{M}^{m/n}$ as

$$\overline{M}^{m/n}(a_0, \phi_0, c_0) = (\overline{M}_1^{m/n}(a_0, \phi_0, c_0), \quad \overline{M}_2^{m/n}(a_0, \phi_0, c_0), \quad \overline{M}_3^{m/n}(a_0, \phi_0, c_0))$$

which we call the subharmonic Melnikov vector. We have the following theorem (see [3] and [4]):

Theorem A Suppose there exists a point (a_0, c_0) such that $\overline{M}_1(a_0, c_0) = 0$, $\overline{M}_3(a_0, c_0) = 0$ and $\partial(\overline{M}_1\overline{M}_3)/\partial(a, c)|_{(a_0, c_0)} \neq 0$. Then, the two-dimensional Poincaré map has an isolated fixed point $(a_0, c_0) + O(\varepsilon^2)$ which corresponds to an isolated periodic orbit for the three-dimensional flow.

In fact we may calculate the components $\overline{M}_1$, $\overline{M}_3$ of the subharmonic Melnikov vector in the original coordinates by the following formula:

$$M_1^{m/n}(\alpha, \theta, c) = \frac{T(\alpha, c)}{2\pi}\Big[\int_0^{mT}\langle \nabla H, g\rangle(q_0^\alpha(t, c), t + \theta)dt - \frac{\partial \hat{H}}{\partial c}(q_0^\alpha(0, c))\int_0^{mT}\langle \nabla C, g\rangle(q_0^\alpha(t, c), t + \theta)dt\Big], \tag{4}$$

$$M_3^{m/n}(\alpha, \theta, c) = \int_0^{mT}\langle \nabla C, g\rangle(q_0^\alpha(t, c), t + \theta)dt. \tag{5}$$

1 Derivation of the Differential Equation Models

The onset of Rayleigh-Benard convection represents a deviation from the basic state of no fluid motion and a linear temperature profile in the vertical direction. In an appropriately scaled Cartesian coordinate system the temperature T at any point in the layer may be written as

$$T - T_0 = -z + T'(x, y, z, t),$$

where T_0 is a reference temperature, z is the vertical coordinate, T' is the deviation from the basic state, and t is time. For convection sufficiently close to onset, T' and the fluid velocity may be expanded in terms of a small parameter ε in the form

$$T' = \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + \cdots.$$

Substituting the above expansion into the appropriate nonlinear equations for the conservation of heat, mass and momentum, at first-order in ε, the equations are linear and separable in the horizon coordinates, so we can write (see Palm[5] and Stuart[2])

$$\begin{bmatrix} T_1 \\ W_1 \end{bmatrix} = \Phi(x,y)\begin{bmatrix} g(z) \\ h(z) \end{bmatrix},$$

where W_1 is the vertical component of velocity at first-order. The function $\Phi(x,y)$ is the plan-form function and satisfies the Helmholtz equation

$$\nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi = -k^2\Phi, \tag{6}$$

where k^2 denotes the sum of the squares of the horizontal wave numbers. Equation (6) has an infinite number of solutions and the actual solution that is observed in given situation cannot be determined from linear theory. In the literature the following particular solutions of (6)

$$\Phi(x,y) = \cos kx, \tag{7}$$

$$\Phi(x,y) = \cos kx + \cos ky, \tag{8}$$

$$\Phi(x,y) = \frac{\cos(\sqrt{3}kx + ky)}{2} + \frac{\cos(\sqrt{3}kx - ky)}{2} + \cos ky \tag{9}$$

are called respectively the roll, square and hexagonal planforms. These are steady solutions and which of them is realized in an experiment is determined by factors such as boundary conditions and properties of the convection fluid.

When the boundaries of the convection layer are very poor conductors of heat, the horizontal lengthscale is so large that an expansion scheme can be developed which separates the vertical and horizontal coordinates. From Jerkins[1] we know that at second-order and higher-order in the expansion, higher-order approximations to the temperature and velocity field have been obtained. For example, expanding $\Phi(x,y)$ in terms of a small parameter ε_1 as

$$\Phi(x,y) = \varepsilon_1\Phi_1 + \varepsilon_1^2\Phi_2 + \varepsilon_1^3\Phi_3 + \cdots. \tag{10}$$

For a square planform, we have (see [1] p.454)

$$\Phi_2 = \frac{-\xi(\cos 2kx + \cos 2ky)}{9} - 2\xi\cos kx\cos ky, \tag{11}$$

$$\Phi_3 = \frac{(15/7 + \xi^2)(\cos 3kx + \cos 3ky)}{128} + \frac{(5/7 + 14\xi^2/9)(\cos 2ky + \cos 2kx)}{16}, \tag{12}$$

where ξ is an $O(1)$ parameter which is a measure of the temperature dependence of viscosity. The expression for Φ_1 is the plan form given by (7), (8) and (9).

In order to discuss the geometrical pattern associated with a given solution of the conviction equations, we need to consider the following equations

$$\frac{dx}{U} = \frac{dy}{V} = \frac{dz}{W} = dt, \tag{13}$$

where U, V are horizontal components of velocity, W is corresponding vertical components. So-called streamlines are the solutions of (13) with $dt \equiv 0$, i.e. with time t as a parameter. For a stationary field of velocity, streamlines coincide with the trajectories of liquid particles. From Stuart[2] and Zaslavsky, et al.[6], we know that

$$U = \frac{1}{k^2} h'(z) \frac{\partial \Phi}{\partial x}, \tag{14}$$

$$V = \frac{1}{k^2} h'(z) \frac{\partial \Phi}{\partial y}. \tag{15}$$

Thus we have the streamline equations

$$\frac{\mathrm{d}x}{\partial \Phi / \partial x} = \frac{\mathrm{d}y}{\partial \Phi / \partial y} = \frac{h'(z)\mathrm{d}z}{k^2 \Phi(x,y) h(z)} = \frac{h'(z)}{k^2}\mathrm{d}t, \tag{16}$$

which defines the spatial pattern of the stream lines.

Let us choose more appropriate coordinate axes and use (8), (11) and (16), so that we obtain the following differential system

$$(\nabla_\varepsilon) \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -\sin x \cos y(\cos z + 2\varepsilon a \cos 2z) - 4\varepsilon b \cos 2x \sin 2y \cos 2z, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\cos x \sin y(\cos z + 2\varepsilon a \cos 2z) - 4\varepsilon b \sin 2x \cos 2y \cos 2z, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = \cos x \cos y(\sin z + a\varepsilon \sin 2z) + \varepsilon b \cos 2x \cos 2y \sin 2z. \end{cases}$$

System (∇_ε) defines the dynamics of a passive ingredient in the field of velocity:

$$\Phi(x,y,z) = \cos x \cos y(\sin z + a\varepsilon \sin 2z) + \varepsilon b \cos 2x \cos 2y \sin 2z. \tag{17}$$

The picture of streamlines in square cells with the second-order approximation in parameter ε will be discussed in the next sections.

2 The Flow of the Three-Dimensional Square Cell Pattern

We first consider the system ($\nabla_{\varepsilon=0}$). It is easy to see that by identifying the planes $x = -\pi$ and $x = \pi$, $y = -\pi$ and $y = \pi$ and $z = -\pi$ and $z = \pi$, the system ($\nabla_{\varepsilon=0}$) can be seen as a system defined on a three-dimensional torus T^3. The planes $x = 0, \pm\pi$, $y = 0, \pm\pi$, $z = 0, \pm\pi$, $y = x$ and $y = -x$ are invariant planes of ($\nabla_{\varepsilon=0}$). We only need to investigate the dynamical behaviour of (∇_ε) in a trianglar prism.

Note that ($\nabla_{\varepsilon=0}$) is a generalized Hamiltonian system, since

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & \sin x \sin y \cos z & -\sin x \cos y \sin z \\ -\sin x \sin y \cos z & 0 & 0 \\ \sin x \cos y \sin z & 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\cos x}{\sin x} \\ \dfrac{-\cos y}{\sin y} \\ 0 \end{pmatrix},$$

where the Hamiltonian and Casmir functions are respectively

$$\widetilde{H}(x,y,z) = \ln \frac{\sin x}{\sin y} = h^* \tag{18}$$

and

$$\widetilde{C}(x,y,z) = \sin y \sin z = c^*. \tag{19}$$

By using the transformation $(\cos x, \cos y, \cos z) \to (u, v, w)$, the system (∇_ε) becomes the following system (denoted by $\bar{\nabla}_\varepsilon$):

$$\dot{u} = (1 - u^2)vw + 2\varepsilon av(1 - u^2)(2w^2 - 1) + 8\varepsilon bv(2u^2 - 1)(2w^2 - 1)\sqrt{(1 - u^2)(1 - v^2)},$$
$$\dot{v} = uw(1 - v^2) + 2\varepsilon au(1 - v^2)(2w^2 - 1) + 8\varepsilon bu(2v^2 - 1)(2w^2 - 1)\sqrt{(1 - u^2)(1 - v^2)},$$
$$\dot{w} = -uv(1 - w^2) - 2\varepsilon auvw(1 - \omega^2) - 2\varepsilon bw(2u^2 - 1)(2v^2 - 1)(1 - w^2).$$

which has Hamiltonian and Casmir functions

$$H(u, v, w) = \ln\left(\frac{1 - v^2}{1 - u^2}\right) = 2h^* \tag{20}$$

and

$$C(u, v, w) = (1 - v^2)(1 - w^2) = c \tag{21}$$

respectively, where $c = (c^*)^2$ and we write $h = \exp(-2h^*)$. On the symplectic leaf $C(u, v, w) = c_0$, $(\tilde{\nabla}_\varepsilon)$ can be reduced to the following slowly varying system (denoted by Θ_ε):

$$\dot{u} = w(1 - u^2)\left(1 - \frac{c_0}{1 - w^2}\right)^{1/2} + \varepsilon\left[2a(1 - u^2)(2w^2 - 1)\left(1 - \frac{c_0}{1 - w^2}\right) + 8b(2u^2 - 1)(2w^2 - 1)(1 - u^2)^{1/2}\left(1 - \frac{c_0}{1 - w^2}\right)^{1/2}\left(\frac{c_0}{1 - w^2}\right)^{1/2}\right],$$
$$\dot{w} = -u(1 - w^2)\left(1 - \frac{c_0}{1 - w^2}\right)^{1/2} - \varepsilon\left[2auw(1 - w^2)\left(1 - \frac{c_0}{1 - w^2}\right)^{1/2} + 2bw(2u^2 - 1)(1 - w^2)\left(1 - \frac{2c_0}{1 - w^2}\right)\right],$$
$$\dot{c} = \varepsilon\left\{4auc_0\left(1 - \frac{c_0}{1 - w^2}\right)^{1/2}(1 - w^2) + 4b\left(1 - \frac{2c_0}{1 - w^2}\right)(c_0(2u^2 - 1)w^2 - 4u(1 - w^2)(2w^2 - 1)\left(1 - \frac{c_0}{1 - w^2}\right)^{1/2}(1 - u^2)^{1/2}\left(\frac{c_0}{1 - w^2}\right)^{1/2})\right\},$$

where $\varepsilon > 0$ sufficiently small and $ab \neq 0$. From $(\tilde{\nabla}_{\varepsilon=0})$, we know that $u = \pm 1$, $v = \pm 1$ and $w = \pm 1$ are invariant planes of the unperturbed vector field. So we consider the case of $0 \leqslant |u| < 1$, $0 \leqslant |v| < 1$ and $0 \leqslant |w| < 1$. We now consider the unperturbed system $(\Theta_{\varepsilon=0})$ on the unit square $\{(u, w) \in \mathbf{R}^2 : 0 \leqslant |u| < 1, 0 \leqslant |w| < 1\}$ of the (u, w)-plane, the origin $(0, 0)$ is a center. For each fixed $c = c_0$, the curve defined by the algebraic equation $w = \pm\sqrt{1 - c_0}$ $(0 < c_0 < 1)$, are two singular straight lines. Therefore, we only can discuss the case of $|w| < \sqrt{1 - c_0}$.

For the system $(\Theta_{\varepsilon=0})$, there is a family of periodic orbit given by the level curves

$$(1 - w^2)(1 - u^2) = c_0 h, \tag{22}$$

where $1 < h < \frac{1}{c_0}$, for fixed $c = c_0$, and with the following parametric representation

$$u(t, k) = \pm\left[1 - \frac{c_0 h}{1 - (1 - c_0 h)\mathrm{sn}^2(\Omega t, k)}\right]^{1/2},$$

$$w(t,k) = \pm\left[1 - \frac{c_0 h}{1 - u^2(t,k)}\right]^{1/2},$$

where $\Omega^2 = h^2(1-c_0)$, $k^2 = (1-c_0h)/(1-c_0)$. $\operatorname{sn}(u,k)$ is a Jacobia elliptic function with elliptic modulus k. The periods of the periodic orbits are

$$T = \frac{2K(k)}{\Omega},$$

where $K(k)$ is the complete elliptic integral of the first kind.

On the basis of the Theorem A of Section 1 by using the relations

$$w^2 = 1 - \frac{c_0 h}{1-u^2}, \quad v^2 = 1 - (1/h) + u^2/h,$$

we have form (4) and (5) that

$$M_1(a,b,c_0,h) = \frac{8bT(1-h)}{\pi h}\left(\int_0^T \sqrt{(h-1+u^2)u^2}\,dt - 2hc_0\int_0^T \frac{\sqrt{(h-1+u^2)u^2}}{(1-u^2)}dt\right) = \frac{8bT(1-h)}{\pi h}(I_0 - 2hc_0 I_{-1}),$$

$$M_3(a,b,c_0,h) = (4ac_0^2\sqrt{h} - 32bc_0h)\int_0^T \frac{\sqrt{(h-1+u^2)u^2}}{(1-u^2)}dt + (8bc_0 + 16bc_0^2)\int_0^T u^2 dt + 16bc_0(1+4c_0)\int_0^T \sqrt{(h-1+u^2)u^2}\,dt - \frac{16bc_0}{h}\int_0^T (1-u^2)u^2 dt - \frac{32bc_0}{h}\int_0^T \sqrt{(h-1+u^2)u^2}(1-u^2)dt - 4bhc_0\int_0^T \frac{u^2}{1-u^2}dt + (4bhc_0^2 - 8bc_0^2)T = n_1 I_{-1} + n_2 I_0 - 2n_3 I_1 + n_4 J_0 - n_5 J_{-1} - n_3 J_1 + n_6,$$

where

$$n_1 = 4ac_0^2\sqrt{h} - 32bc_0h, \quad n_2 = 16bc_0(1+4c_0), \quad n_3 = \frac{16bc_0}{h},$$

$$n_4 = 8bc_0 + 16bc_0^2, \quad n_5 = 4bhc_0, \quad n_6 = T(4bhc_0^2 - 8bc_0^2);$$

$$I_i = \int_0^T \sqrt{(h-1+u^2)u^2}(1-u^2)^i dt,$$

$$J_i = \int_0^T (1-u^2)^i u^2 dt, \qquad i = -1, 0, 1.$$

Letting $M_1 = 0$, we have

$$I_0 = 2hc_0 I_{-1}. \tag{23}$$

When $I_0 = 2hc_0 I_{-1}$, letting $M_3 = 0$, we have

$$m_1 I_{-1} - 2m_2 I_1 + m_3 J_0 - m_4 J_{-1} - m_2 J_1 + m_5 = 0, \tag{24}$$

where

$$m_1 = 4ac_0^2\sqrt{h} + 128bhc_0^3, \quad m_2 = \frac{16bc_0}{h},$$

$$m_3 = 8bc_0 + 16bc_0, \quad m_4 = 4bhc_0^2, \quad m_5 = (4bhc_0^2 - 8bc_0^2)T.$$

Therefore if there exists (a_0, b_0, c_0, h) for $c_0 \in (0,1)$ and $h \in (1, 1/c_0)$ such that (23) and (24) hold, then the system (∇_ε) has a periodic solution provided that the condition $\partial(M_1, M_3)/\partial(c_0, h) \neq 0$ is satisfied, and ε is sufficiently small. From Theorem A, we have the following conclusion:

Theorem B If there exists (a_0, b_0, c_0, h), $c_0 \in (0,1)$ and $h \in (1, 1/c_0)$ such that (23) and (24) hold and $\partial(M_1, M_3)/\partial(c_0, h)\,|_{(a_0, b_0, c_0, h)} \neq 0$, then for sufficiently small ε, the system (∇_ε) has a periodic solution on the sympletic leaf $C(u, v, w) = c_0$.

The Theorem B gives rise to the existence of periodic stream lines of the system (17), which coincide with the experimental results of [1].

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