The Predictability Problems in Numerical Weather and Climate Prediction①

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ABSTRACT

The uncertainties caused by the errors of the initial states and the parameters in the numerical model are investigated. Three problems of predictability in numerical weather and climate prediction are proposed, which are related to the maximum predictable time, the maximum prediction error, and the maximum admissible errors of the initial values and the parameters in the model respectively. The three problems are then formulated into nonlinear optimization problems. Effective approaches to deal with these nonlinear optimization problems are provided. The Lorenz' model is employed to demonstrate how to use these ideas in dealing with these three problems.

Key words: Predictability, Weather, Climate, Numerical model, Optimization

1. Introduction

The predictability problem in numerical weather and climate prediction is concerned with the uncertainties of the forecast, which is usually classified into two types. The first kind of predictability is related to the initial errors, and the second kind of predictability is to the model errors. At present, it is generally accepted in the society of the atmospheric sciences that the first kind of predictability is a main problem in numerical weather prediction, although the model errors in some models, e.g. the mesoscale model, have not been successfully solved yet. For short–term numerical climate prediction, the second kind of predictability is considered to be the main problem. We should also point out that for the prediction problem of interannual climate variability such as ENSO (El Niño and Southern Oscillation), the first kind of predictability has also been receiving increasingly attention (Thompson 1998; Xue et al. 1997; Yuan et al. 2000).

The definition of the model error varies with the authors, in this paper, we adopt the following definition (see Talagrand 1997): If the initial value of the model is the true state, then the difference between the values of the forecast and the true state at the prediction time is called the model error.

From the above definition of the model error, it is easily seen that there are many factors causing model errors, for example, ignoring some physical processes, the errors in the

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parameters of the model which includes the formulation of the forcing, discretization error of the model, and even the round–off error (Li et al. 2000), etc.

Practically it is difficult to distinguish the first kind of predictability from the second kind of predictability. And it is almost impossible to study the model error completely. The main purpose of this paper is to consider the uncertainty of the prediction caused by the initial error and the parameter error in the model, which is generally considered to be the main problem in the model error. The explicit meaning of this is explained as follows. Denote M_t the propagator which propagates the state from the initial time to time t, $\boldsymbol{u}_0$ is the initial value, μ is the parameter, $\boldsymbol{u}_t = M_t(\boldsymbol{u}_0, \mu)$. $\boldsymbol{u}_0^t$ and $\boldsymbol{u}_t^t$ are the true values of the state at initial time and time t respectively, μ^t is the true value of parameter, throughout this paper we assume

$$\boldsymbol{u}_t^t = \boldsymbol{u}_t = M_t(\boldsymbol{u}_0^t, \mu^t) , \tag{1.1}$$

that is to say, the prediction error is only caused by the initial error and parameter error. In this paper, we will first classify three problems of the predictability in the numerical weather and climate prediction and reduce them into nonlinear optimization problems respectively. Then how to deal with these problems by using the information on the errors of the initial value and the parameter are investigated. Finally, we use Lorenz′ model (Lorenz 1965a, b) to show how to realize the above idea in the research of predictability.

2. Three problems of predictability

On the basis of pratical demands the study of the predictability of numerical weather and climate prediction can be classified into three problems.

Problem 1. Assume that the initial observation $\boldsymbol{u}_0^{\text{obs}}$ and the first given value of the parameter μ^g are known. At prediction time T, the maximum allowing prediction error in terms of the norm $\|\cdot\|_A$ measuring $\boldsymbol{u}$ is

$$\| M_T(\boldsymbol{u}_0^{\text{obs}}, \mu^g) - \boldsymbol{u}_T^t \|_A \leqslant \varepsilon , \tag{2.1}$$

where $\boldsymbol{u}_T^t$ is the true value of the state at time T. It is our purpose to find out the maximum predictable time T_ε under the above conditions. This problem can be reduced to a nonlinear optimization problem:

$$T_\varepsilon = \max\{\tau \mid \| M_t(\boldsymbol{u}_0^{\text{obs}}, \mu^g) - \boldsymbol{u}_t^t \|_A \leqslant \varepsilon, \quad 0 \leqslant t \leqslant \tau\} . \tag{2.2}$$

Since the true value $\boldsymbol{u}_t^t$ cannot be obtained exactly, so it is impossible to obtain the exact value of T_ε by solving this nonlinear optimization problem. However, if we know more information about the errors of initial value and the parameters, useful estimation on T_ε can be derived by using some methods. For example, assume that the errors of the initial value and the first given value of the parameter are known as follows

$$\| \boldsymbol{u}_0^t - \boldsymbol{u}_0^{\text{obs}} \|_A \leqslant \delta_1 , \quad \| \mu^t - \mu^g \|_B \leqslant \delta_2 , \tag{2.3}$$

where $\|\ \ \|_B$ is a norm of μ measuring the parameters in the model. Then we can investigate the following nonlinear optimization problem

$$T_{\varepsilon l} = \min_{u_0 \in B_{\delta_1}, \mu \in B_{\delta_2}} \{ T_{u_0,\mu} | T_{u_0,\mu} = \max \tau ,$$

$$\| M_t(\boldsymbol{u}_0, \mu) - M_t(\boldsymbol{u}_0^{obs}, \mu^g) \| \leqslant \varepsilon, \quad 0 \leqslant t \leqslant \tau \} , \tag{2.4}$$

where B_{δ_1}, B_{δ_2} are the balls with centers at $\boldsymbol{u}_0^{obs}$, μ^g and radius δ_1, δ_2 respectively.

It is not difficult to prove that

$$T_{\varepsilon l} \leqslant T_\varepsilon . \tag{2.5}$$

From (2.4), we have the lower bound of the maximum predictable time. It is also clear that the more accurate the initial observation $\boldsymbol{u}_0^{obs}$ and the parameter μ, the more accurate this lower bound.

Remark: In the operational numerical weather and climate prediction, the observational data are seldom used as the initial values to the numerical model. Instead, the analysis fields, which are yielded by the initialization and data assimilation processes, are adopted as the initial values. To make this paper more readable, we use the term "observation" rather than "analysis" in this paper.

Problem 2. Suppose that the initial observation $\boldsymbol{u}_0^{obs}$ and the first given values of the parameter μ^g are known, for a given prediction time T, look for the prediction error, i. e., find out

$$E = \| M_T(\boldsymbol{u}_0^{obs}, \mu^g) - \boldsymbol{u}_T^t \|_A . \tag{2.6}$$

Similar to the above problem, since the true value $\boldsymbol{u}_T^t$ cannot be obtained exactly, it is also impossible to get the exact value of E. But if some information on errors of the initial value and the parameter are available , e.g. (2.3) holds, then we can consider the nonlinear optimization problem

$$E_u = \max_{u_0 \in B_{\delta_1}, \mu \in B_{\delta_2}} \| M_T(\boldsymbol{u}_0, \mu) - M_T(\boldsymbol{u}_0^{obs}, \mu^g) \|_A . \tag{2.7}$$

Without much difficulty, we can prove that

$$E \leqslant E_u . \tag{2.8}$$

In this way we establish the upper bounds on the prediction error, the accuracy of these estimations depends on those of the initial value and the parameters in the model too.

Problem 3. Assume that the initial observation $\boldsymbol{u}_0^{obs}$, the first given value of the parameter μ^g are available. At the prediction time T, the allowing maximum prediction error is (2.1). Our purpose is to determine the allowing maximum initial error and the parameter error. More precisely, look for the maximum δ, such that if (2.3) holds with $\delta = \delta_1 + \delta_2$, then (2.1) holds.

This problem can also be reduced to an optimization problem as follows:

$$\delta_{max} = \max_{\delta} \{ \delta | \ \| \boldsymbol{u}_0^t - \boldsymbol{u}_0^{obs} \|_A \leqslant \delta_1, \ \| \mu^t - \mu^g \|_B \leqslant \delta_2 ,$$

$$\text{if} \quad \delta_1 + \delta_2 = \delta, \quad \text{then} \quad \| M_T(\boldsymbol{u}_0^{obs}, \mu^g) - \boldsymbol{u}_T^t \|_A \leqslant \varepsilon \} . \tag{2.9}$$

Following the above idea, we can estimate δ. Investigating the problem

$$\bar{\delta}_{max} = \max_{\delta}\{\delta | \ \| M_T(u_0^{obs}, \mu^g) - M_T(u_0, \mu)\|_A \leqslant \varepsilon ,$$

$$u_0 \in B_{\delta_1} , \quad \mu \in B_{\delta_2} , \quad \delta_1 + \delta_2 = \delta\} , \tag{2.10}$$

we can conclude that

$$\bar{\delta}_{max} \leqslant \delta_{max} . \tag{2.11}$$

Remark: In the above problems, if the errors in the parameter can be ignored, and furthermore the model can be assumed to be perfect, the problems are the three ones of the first kind of predictability; on the other hand, if there exists no initial error, the problems become those of the second kind of predictability concerning the parameter error.

3. An example–three problems with the first kind of predictability of Lorenz system

In this section, we study the three problems of predictability of Lorenz system as an example. For simplicity, we assume that the model is perfect.

The Lorenz system consists of a set of three ordinary differential equations:

$$\begin{cases} \dfrac{dx}{dt} = -\sigma x + \sigma y , \\ \dfrac{dy}{dt} = -xz + rx - y , \\ \dfrac{dz}{dt} = xy - bz , \\ (x,y,z)|_{t=0} = (x_0, y_0, z_0) . \end{cases} \tag{3.1}$$

where σ, r and b are the parameters. Here we adopt the Lorenz' original choice of parameter values: $\sigma = 10$, $r = 28$, $b = 8/3$ (Lorenz 1965).

It is easy to find out that Lorenz system has three stationary points:

$$\begin{cases} O: \ (x,y,z) = (0,0,0) \\ C_1: \ (x,y,z) = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \\ C_2: \ (x,y,z) = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) . \end{cases} \tag{3.2}$$

We choose these three stationary points as initial observations to study the three problems with the first kind of predictability of the system. It should be pointed out that other points could be adopted as the initial observations too. The system is integrated by middle point scheme with time step $dt = 0.01$.

3.1 *The first predictability problem of Lorenz system*

Let M be the propagator of system (3.1), O^* the initial observation, $X = (x_0, y_0, z_0)$, and $u_0 = O^* + X$. Assume that the initial observational error in terms of a chosen norm $\| \cdot \|$ is not larger than δ. With these notations, $u_0 \in B_\delta$ (see section 2) is equivalent to $\|X\| \leqslant \delta$. The lower bound of the maximum predictable time given by (2.4) under the above conditions is

$$T_{\varepsilon l} = \min_{\|X\| \leqslant \delta}\{T_X | T_X = \max\tau \ , \ \|M_t(O^* + X) - M_t(O^*)\| \leqslant \varepsilon \ , \ 0 \leqslant t \leqslant \tau\} \ . \quad (3.3)$$

In this paper, two norms $\|X\|_2 = \sqrt{x_0^2 + y_0^2 + z_0^2}$ and $\|X\|_\infty = \max\{|x_0|, |y_0|, |z_0|\}$ are employed to measure the errors. For different initial observational error bounds $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$, we adopt the following algorithms to obtain the lower bound of maximum predictable time $T_{\varepsilon l}$. Firstly for the domain $\|X\|_\infty \leqslant \delta$, we take cube mesh of side 0.01, then integrate the model from each mesh–point and obtain the maximum predictable time T_X for this point. The minimum of all these is what we require. Of course the result depends on the mesh we utilized. In the calculation, we have tried several different meshes, and find that there is no essential difference if the side of the mesh is not larger than 0.01. So we use 0.01 in the calculation.

Secondly for the ball $\|X\|_2 \leqslant \delta$, we consider its circumscribed cube. For any mesh points outside the ball, we connect this point with the center of the ball, and take the intersection point of this line with the surface of the ball, then integrate the model from each of these intersection points and the mesh points inside the ball. Similar to the case of $\|X\|_\infty \leqslant \delta$, the results are obtained.

The details of the results for initial observation O are shown in Tables 1 and 2, where δ is the initial observational error bound and $T_{\varepsilon l}$ is the lower bound of the maximum predictable time with the maximum allowing prediction error ε. Here $T_{\varepsilon l}$ is the number of the time steps of the numerical integration.

Table 1. $T_{\varepsilon l}$ for initial observation O with $\|X\|_\infty \leqslant \delta$

T \ δ / ε	0.005	0.01	0.05	0.1	0.15	0.2
0.6	76	64	36	26	17	13
1.0	85	73	45	33	26	21
1.4	91	79	51	39	32	27
1.8	96	84	56	43	36	31
2.2	99	87	59	47	40	35

Table 2. $T_{\varepsilon l}$ for initial observation O with $\|X\|_2 \leqslant \delta$

T \ δ / ε	0.005	0.01	0.05	0.1	0.15	0.2
0.6	81	69	41	29	22	17
1.0	90	78	50	38	30	25
1.4	95	83	55	43	36	31
1.8	100	88	60	48	41	36
2.2	103	91	63	51	44	39

Obviously if $(x(t), y(t), z(t))$ is a solution to Lorenz system, $(-x(t), -y(t), z(t))$ is a solution to that too. Due to this symmetric property, it is easy to know that the lower bounds of the maximum predictable time of the initial observation C_1 and C_2 are equivalent. For simplicity, we only show the results of C_1 in Tables 3 and 4.

Table 3. $T_{\varepsilon l}$ for initial observation C_1 with $\|X\|_\infty \leqslant \delta$

ε \ δ (T)	0.005	0.01	0.05	0.1	0.15	0.2
0.4	2204	1812	909	516	290	145
0.8	2614	2244	1318	907	701	516
1.2	2861	2431	1527	1133	907	762
1.6	2982	2614	1527	1317	1072	907
2.0	3112	2738	1812	1442	1195	1035

Table 4. $T_{\varepsilon l}$ for initial observation C_1 with $\|X\|_2 \leqslant \delta$

ε \ δ (T)	0.005	0.01	0.05	0.1	0.15	0.2
0.4	2367	1991	1069	694	450	271
0.8	2745	2366	1445	1068	826	693
1.2	2985	2609	1687	1312	1068	889
1.6	3164	2743	1867	1444	1249	1068
2.0	3288	2867	1991	1568	1373	1192

It is clear from Tables 1–4 that the lower bound of maximum predictable time of the initial observation C_1 is much longer than the corresponding one of the initial observation O. This indicate that there exists stronger predictability around C_1.

Fig. 1 is $T_{\varepsilon l}$ for initial observation O with $\|X\|_2 \leqslant \delta$. Fig. 2 is $T_{\varepsilon l}$ for initial observation C_1 with $\|X\|_2 \leqslant \delta$.

3.2. *The second predictability problem of Lorenz system*

Denote M, O^* and X as above, if the initial observational error in terms of a chosen norm $\|\cdot\|$ is not larger than δ, the upper bound of prediction error of the initial observation O^* at time T is

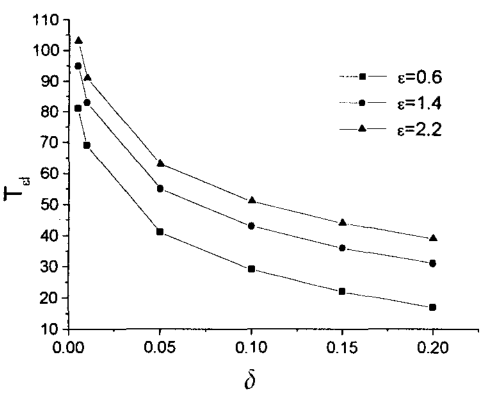

Fig. 1. $T_{\varepsilon l}$ for initial observation O with norm $\|\ \|_2$.

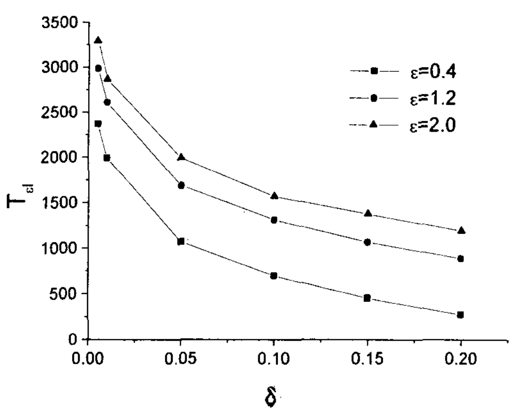

Fig. 2. $T_{\varepsilon l}$ for initial observation C_1 with norm $\|\ \|_2$.

$$E_u = \max_{\|X\| \leqslant \delta} \| M_T(O^* + X) - M_T(O^*) \| . \tag{3.4}$$

If the initial perturbation $X_\delta^* = (x_{0\delta}^*, y_{0\delta}^*, z_{0\delta}^*)$ superposed on O^* satisfies

$$E_u = \| M_T(O^* + X_\delta^*) - M_T(O^*) \| , \tag{3.5}$$

it is called conditional optimally growing perturbation of O^*. The value

$$\lambda_\delta = \frac{E_u}{\delta}$$

represents the maximum rate of the evolution of the initial perturbations superposed on the observation.

In the following numerical experiments, the optimization algorithms adopted are limited memory BFGS method (Liu and Nocedal 1989) for norm $\| \cdot \|_\infty$, which is an extension of the conjugate gradient method. And for norm $\| \cdot \|_2$, the optimization algorithm is the trust region method (Yuan 1990).

3.2.1 *The prediction error of the initial observation O*

For two different norms $\|X\|_2 = \sqrt{x_0^2 + y_0^2 + z_0^2}$ and $\|X\|_\infty = \max\{|x_0|, |y_0|, |z_0|\}$, and $T = 20, 30$ time steps, we obtained the upper bounds of the prediction error of the initial observation O, the conditional nonlinear optimally growing perturbations (CNOGPs) and the maximum evolution rates of CNOGPs numerically. It is found that for initial observational error bound $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$, $\delta \in [0.08, 4.0]$, there are two CNOGPs for $T = 20$ time steps respectively, which are only differ by the signs of the x and y components and correspond to the same maximum growing rate λ_δ. It is clear from the symmetry of Lorenz model that if (x, y, z) is a CNOGP, $(-x, -y, z)$ is also a CNOGP. Our numerical results do verify this. The results for $T = 20$ with error bound $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$ are shown in Tables 5 and 6, where E_u is the upper bound of prediction error, $X_\delta^* = (x_{0\delta}^*, y_{0\delta}^*, z_{0\delta}^*)$ is one of the two CNOGPs respectively. For simplicity, the results of another CNOGP is not shown here.

Table 5. E_u, λ_δ and X_δ^* for $T = 20$ with $\|X\|_2 \leqslant \delta$

δ	0.08	0.8	1.6	2.4	3.2	4.0
E_u	0.2828	2.8272	5.6501	8.4644	11.2658	14.0502
λ_δ	3.5349	3.5340	3.5313	3.5268	3.5206	3.5126
$x_{0\delta}^*$	6.2549×10^{-2}	0.6254	1.2503	1.8743	2.4967	3.1171
$y_{0\delta}^*$	4.9876×10^{-2}	0.4986	0.9967	1.4936	1.9887	2.4812
$z_{0\delta}^*$	-1.2695×10^{-4}	-1.4073×10^{-2}	-5.6623×10^{-2}	−0.1277	−0.2274	−0.3561

Table 6. E_u, λ_δ and X_δ^* for $T = 20$ with $\|X\|_\infty \leqslant \delta$

δ	0.08	0.8	1.6	2.4	3.2	4.0
E_u	0.3607	3.6833	7.4995	11.3888	15.2860	19.1216
λ_δ	4.5097	4.6041	4.6871	4.7453	4.7769	4.7804
$x_{0\delta}^*$	−0.0800	−0.800	−1.6000	−2.4000	−3.2000	−4.0000
$y_{0\delta}^*$	−0.0800	−0.800	−1.6000	−2.4000	−3.2000	−4.0000
$z_{0\delta}^*$	−0.0800	−0.800	−1.6000	−2.4000	−3.2000	−4.0000

Similar to the case of $T=20$, there exist two symmetrical CNOGPs for $T=30$ with $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$ respectively. Tables 7 and 8 show the results of one of the two CNOGPs for initial observational error bounds $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$ respectively.

Table 7. E_u, λ_δ and X_δ^* for $T=30$ with $\|X\|_2 \leqslant \delta$

δ	0.08	0.8	1.6	2.4	3.2	4.0
E_u	0.5048	5.0408	10.0394	14.9548	19.7485	24.3850
λ_δ	6.3097	6.3010	6.2746	6.2312	6.1714	6.0963
$x_{0\delta}^*$	6.3015×10^{-2}	0.6300	1.2587	1.8851	2.5078	3.1251
$y_{0\delta}^*$	4.9286×10^{-2}	0.4927	0.9843	1.4736	1.9595	2.4406
$z_{0\delta}^*$	-2.0179×10^{-4}	-2.0518×10^{-2}	-8.2388×10^{-2}	-0.1863	-0.3338	-0.5265

Table 8. E_u, λ_δ and X_δ^* for $T=30$ with $\|X\|_\infty \leqslant \delta$

δ	0.08	0.8	1.6	2.4	3.2	4.0
E_u	0.6458	6.5926	13.2358	19.5124	25.0001	29.3198
λ_δ	8.0735	8.2407	8.2724	8.1302	7.8126	7.3299
$x_{0\delta}^*$	-0.0800	-0.800	-1.6000	-2.4000	-3.2000	-4.0000
$y_{0\delta}^*$	-0.0800	-0.800	-1.6000	-2.4000	-3.2000	-4.0000
$z_{0\delta}^*$	-0.0800	-0.800	-1.6000	-2.4000	-3.2000	-4.0000

Fig. 3 is X_δ^* for $T=30$ with $\|X\|_2 \leqslant \delta$. Fig. 4 is λ_δ for $T=30$ with $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$. Fig. 5 is E_u for $T=30$ with $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$.

3.2.2 *The prediction error of the initial observations* C_1 and C_2

We have pointed out that if $(x(t), y(t), z(t))$ is a solution of (3.1), $(-x(t), -y(t), z(t))$ is a solution of (3.1) too. Since C_1 and C_2 only differ by the signs of their x and y components, we derived that the CNOGPs of the initial observations C_1 and C_2 only differ by the signs of their x and y components for the same prediction error bound. For simplicity, in the following we only consider the case of initial observation C_1.

It is different from the case of initial observation O for initial observation C_1, there exists a CNOGP for $T=50$, 80 time steps respectively. Table 9 is the upper bounds of the

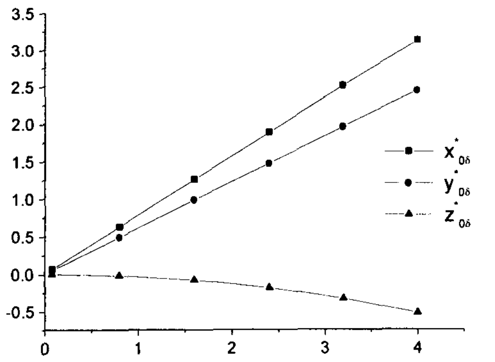

Fig. 3. X_δ^* for $T=30$ with norm $\|\ \|_2$ in case of initial observation O.

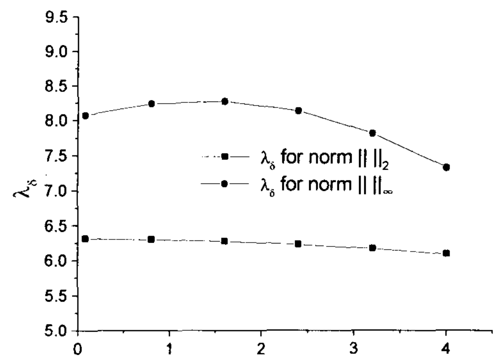

Fig. 4. λ_δ for $T=30$ with norm $\|\ \|_2$ and $\|\ \|_\infty$ in case of initial observation O.

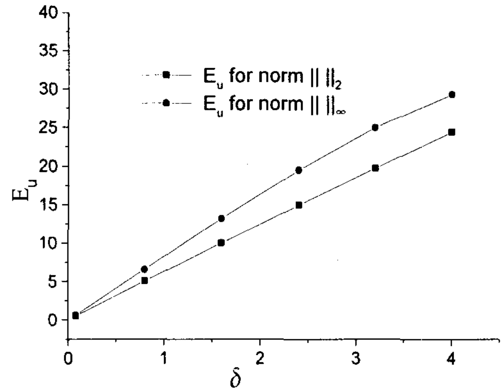

Fig. 5. E_u for $T=30$ with norm $\| \ \|_2$ and $\| \ \|_\infty$ in case of initial observation O.

prediction error E_u, the CNOGP $X^*_{0\delta}$ and the maximum evolution rate of the CNOGP λ_δ for time steps $T=50$ with initial observational error bound $\|X\|_2 \leqslant \delta$. Table 10 is for $T=80$'s.

Table 9. E_u, λ_δ and X^*_δ for $T=50$ with $\|X\|_2 \leqslant \delta$

δ	0.08	0.8	1.6	2.4	3.2	4.0
E_u	0.09983	1.1043	2.0669	3.1625	4.3057	5.5012
λ_δ	1.2478	1.2697	1.2918	1.3177	1.3455	1.3753
$x^*_{0\delta}$	−0.00121	0.0038	0.0445	0.1241	0.2439	0.4044
$y^*_{0\delta}$	0.0605	0.6229	1.2838	1.9815	2.7133	3.4755
$z^*_{0\delta}$	0.0523	0.5023	0.9538	1.3484	1.6789	1.9384

For initial observational error bound $\|X\|_\infty \leqslant \delta$, $\delta \in [0.08, 4.0]$, there is also a CNOGP for $T=50,80$ time steps respectively. The details are shown in Tables 11 and 12.

Fig. 6 is X^{*2}_δ for $T=80$ with $\|X\|_2 \leqslant \delta$. Fig. 7 is λ_δ for $T=80$ with $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$. Fig. 8 is E_u for $T=80$ with $\|X\|_2 \leqslant \delta$ and $\|X\|_\infty \leqslant \delta$.

Table 10. E_u, λ_δ and X^*_δ for $T=80$ with $\|X\|_2 \leqslant \delta$

δ	0.08	0.8	1.6	2.4	3.2	4.0
E_u	0.1105	1.1125	2.2402	3.3824	4.5390	5.7098
λ_δ	1.3817	1.3906	1.4001	1.4093	1.4184	1.4275
$x^*_{0\delta}$	−0.02809	−0.2786	−0.5552	−0.8347	−1.1219	−1.4275
$y^*_{0\delta}$	−0.07489	−0.7492	−1.4978	−2.3441	−2.9869	−3.7252
$z^*_{0\delta}$	−0.00179	−0.03292	−0.09157	−0.1652	−0.2445	−0.3214

Table 11. E_u, λ_δ and X^*_δ for $T=50$ with $\|X\|_\infty \leqslant \delta$

δ	0.08	0.8	1.6	2.4	3.2	4.0
E_u	0.1170	1.1619	2.3048	3.4334	4.5565	5.6836
λ_δ	1.4627	1.4523	1.4404	1.4306	1.4239	1.4208
$x^*_{0\delta}$	−0.0800	−0.8000	−1.6000	−2.4000	−3.2000	−4.0000
$y^*_{0\delta}$	−0.0800	−0.8000	−1.6000	−2.4000	−3.2000	−4.0000
$z^*_{0\delta}$	0.0800	0.8000	1.6000	2.4000	3.2000	4.0000

Table 12. E_u, λ_δ and X_δ^* for $T=80$ with $\|X\|_\infty \leqslant \delta$

δ	0.08	0.8	1.6	2.4	3.2	4.0
E_u	0.1157	1.1548	2.2939	3.4109	4.5083	5.5916
λ_δ	1.4468	1.4435	1.4337	1.4252	1.4088	1.3979
$x_{0\delta}^*$	0.0800	0.8000	1.6000	2.4000	3.2000	4.0000
$y_{0\delta}^*$	−0.0800	−0.8000	−1.6000	−2.4000	−3.2000	−4.0000
$z_{0\delta}^*$	−0.0800	−0.8000	−1.6000	−2.4000	−3.2000	−4.0000

3.3 *The third predictability problem of Lorenz system*

Suppose that the parameters in the system are accurate, M, O^* and X are the same as above. For given allowing prediction error ε, at prediction time T, it follows from (2.10) that the lower bound of the allowing maximum initial error is

$$\bar{\delta}_{\max} = \max_{\delta}\{\delta | \ \|M_T(O^* + X) - M_T(O^*)\| \leqslant \varepsilon, \|X\| \leqslant \delta\} \ . \tag{3.6}$$

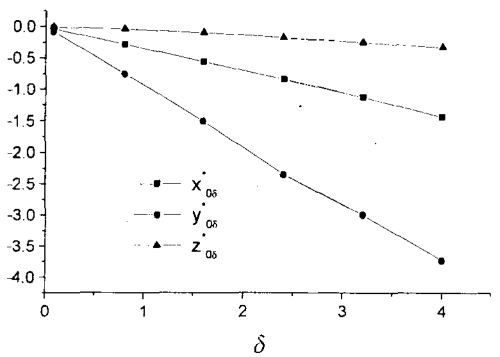

Fig. 6. X_δ^{*2} for $T=80$ with norm $\| \ \|_2$ in case of initial observation C_1.

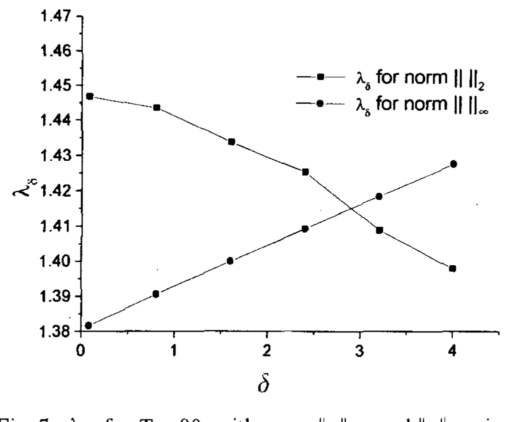

Fig. 7. λ_δ for $T=80$ with norm $\| \ \|_2$ and $\| \ \|_\infty$ in case of initial observation C_1.

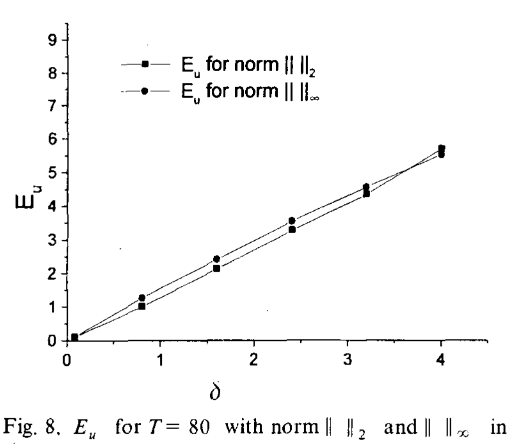

Fig. 8. E_u for $T=80$ with norm $\| \ \|_2$ and $\| \ \|_\infty$ in case of initial observation C_1.

For different prediction time T and allowing maximum prediction errors ε, we adopt the following algorithms to obtain the corresponding allowing maximum initial error bound $\bar{\delta}_{max}$ for norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$. For a first guess δ, we have its corresponding cube in case of norm $\|\cdot\|_\infty$ (the circumscribed cube in case of $\|\cdot\|_2$). Similar to the algorithms in section 3.1, we use cube mesh of side 0.01, and integrate the model from each grid point. If this δ satisfies $\|M_T(O^* + X) - M_T(O^*)\| \leq \varepsilon$, we try another δ larger than this one. Step by step, we finally find the maximum one $\bar{\delta}_{max}$.

The details of the result for initial observation O^* are shown in Tables 13–16.

Table 13. $\bar{\delta}_{max}$ for initial observation O with norm $\|\cdot\|_2$

ε \ $\bar{\delta}_{max}$ \ T	20	40	60	80
0.6	0.1719	0.0597	0.0209	0.0066
1.0	0.2846	0.0901	0.0348	0.0099
1.4	0.3963	0.1270	0.0487	0.0154
1.8	0.5092	0.1638	0.0575	0.0198
2.2	0.6225	0.1977	0.0665	0.0242

Table 14. $\bar{\delta}_{max}$ for initial observation O with norm $\|\cdot\|_\infty$

ε \ $\bar{\delta}_{max}$ \ T	20	40	60	80
0.6	0.1328	0.0418	0.0132	0.0042
1.0	0.2207	0.0695	0.0221	0.0071
1.4	0.3081	0.0973	0.0312	0.0098
1.8	0.3953	0.1250	0.0397	0.0126
2.2	0.4819	0.1528	0.0486	0.0154

Table 15. $\bar{\delta}_{max}$ for initial observation C_1 with norm $\|\cdot\|_2$

ε \ $\bar{\delta}_{max}$ \ T	60	460	860	1260
0.4	0.3483	0.1418	0.0830	0.0421
0.8	0.6889	0.2832	0.1652	0.0794
1.2	1.0240	0.4239	0.2460	0.1090
1.6	1.3527	0.5641	0.3254	0.1401
2.0	1.6739	0.7036	0.4017	0.1794

Table 16. $\bar{\delta}_{max}$ for initial observation C_1 with norm $\|\cdot\|_\infty$

ε \ $\bar{\delta}_{max}$ \ T	60	460	860	1260
0.4	0.2501	0.1499	0.0611	0.0469
0.8	0.4999	0.2259	0.1201	0.0806
1.2	0.7499	0.3493	0.1768	0.0999
1.6	0.9499	0.4499	0.2314	0.1398
2.0	1.1999	0.5499	0.2951	0.1499

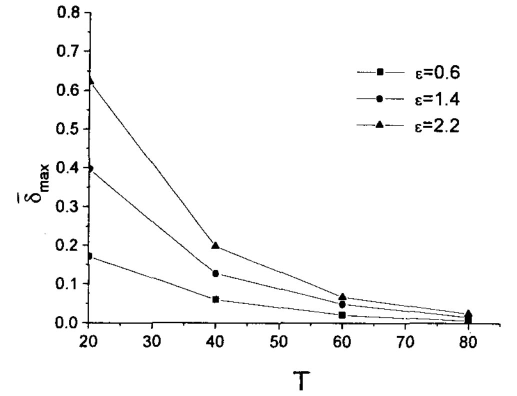

Fig. 9. $\bar{\delta}_{max}$ for initial observation O with norm $\| \ \|_2$.

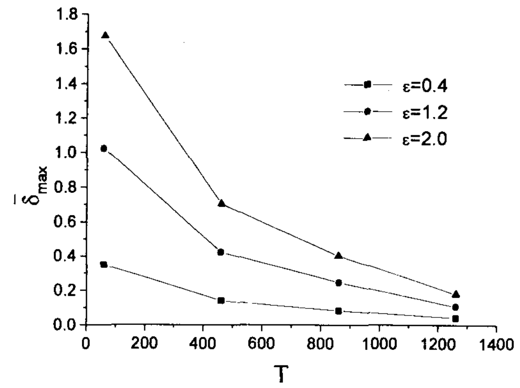

Fig. 10. $\bar{\delta}_{max}$ for initial observation C_1 with norm $\| \ \|_2$.

Tables 13–16 show that for given prediction error, the lower bound of the allowing maximum initial error decreases with the prediction time T. On the other hand, for given prediction time T, the lower bound of the allowing maximum initial error increases with the prediction error.

Fig. 9 is $\bar{\delta}_{max}$ for the initial observation O with norm $\| \cdot \|_2$. Fig. 10 is $\bar{\delta}_{max}$ for initial observation C_1 with norm $\| \cdot \|_2$.

In this section we only use Lorenz model as an example to show how to deal with these three predictability problems formulated in this paper. For much complicated models employed in the numerical weather and climate prediction, the involved numerical optimization problems are challenging, we will discuss this in detail in the following section.

4. Discussion and conclusion

In this paper, we classified the study of predictability of numerical weather and climate prediction into three problems related to maximum prediction time, the maximum prediction error, and the maximum admissible errors of the initial values and of the parameters in the model. All of these problems can be reduced to nonlinear optimization problems. Because the true values of the state of the atmosphere and of the parameters in the model cannot be obtained, so it is impossible to obtain the exact information of the predictability. By utilizing the information about the errors of the initial state and the parameters, we proved that the effective estimation of the predictability can be obtained by solving the corresponding nonlinear optimization problems. As an example, we employed the well–known Lorenz model to investigate these three problems.

Although almost forty years has passed since the pioneer work of Lorenz on the predictability problem (Lorenz 1965a, b), the above three problems for the numerical weather and climate prediction have not been well–studied yet. The main reason could be as follows.

Firstly, the operational numerical weather prediction models at present are of high dimensions, e.g. the dimension of the model used in ECMWF in 1998 is 3×10^7. To solve the above nonlinear optimization problems with such high dimension, the capacity of the existing computers (memory, speed, etc.) could not serve our purpose.

Secondly, the models governing the motions of the atmosphere are nonlinear ones, the parameterization of the physical processes, the formulation of the external forcing, and the constraint conditions related to the errors of the observations and the parameters in the model are complicated too. All these make the suggested three problems become nonlinear, optimization problems with complex constraint conditions. In some cases, the problems are non–smooth one too. At present, computational mathematicians are still attacking them in order to give effective and ripe algorithms.

Thirdly, facing the above–mentioned difficulties, the scientists in atmospheric sciences considered that these problems could only be investigated in the future. Consequently there are little theoretical research on them. For example, the problems such as how to obtain the information on the errors of the observations and the parameters in the models, how to choose proper norms to measure the errors, and how to determine the maximum admissible errors, etc. have not been wellstudied yet.

With the development of the economy and society, meteorologists are required to provide the numerical weather and climate prediction with higher accuracy. Quantifying predictability will be one of the main subjects in the study of the uncertainties of the forecast. Although the above three problems of the predictability concerning the estimations of the uncertainties are reduced to nonlinear optimization problems with high dimensions, it is expected that the rapid development of computers will serve our purpose in the future not too far, and it is time to devote our energies to this study.

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数值天气预报和气候预测的可预报性问题

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摘　　要

考察由初始状态误差和模式中参数误差所引起的预报结果的不确定性。提出了数值天气预报与气候预测中三类可预报性问题，即，最大可预报时间，最大预报误差，初值与参数的最大允许误差。然后将这三类问题化成了对应的非线性优化问题，给出了处理此类非线性优化问题的思路，并且用数值方法对 Lorenz 模型研究了这三类问题。

关键词: 可预报性，天气，气候，数值模式，非线性优化