应 用 数 学
MATHEMATICA APPLICATA
2004,17(4):603～611

Chaotic and Resonant Stream Lines in the Quasi-symmetry Flows

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Abstract: The dynamical systems associated with fluid particle motion of the qth order ($q = 3$ or 6) quasi-symmetry flows are studied by using the Melnikov method. It is suggested that the quasi-symmetry flows ($q = 3$ or 6) have chaotic and resonant stream lines under proper conditions, which, analytically, is given in this paper.

Key words: Resonant stream lines; Chaotic stream lines; Smale horse shoes; Heteroclinic cycles

CLC Number: O175.14 **AMS(2000)Subject Classification**: 34C23

Document code: A **Article ID**: 1001-9847(2004)04-0603-09

1. Introduction

Turbulence is understood to be a chaotic phenomenon in hydrodynamics according to the point of view of nonlinear dynamics. People hope that the chaotic trajectories of a real fluid motion can be found in the physical Euclidean space rather than in the phase space. The current booming development of the works in the field of Lagrangian turbulence (or chaotic advection) belongs to this kind of efforts[1,8,9]. So called the Lagrangian turbulence, it means that the stream lines are arranged chaotically in Euclidean space, where the famous ABC flow [2,12] is a standard example of the Lagrangian picture.

If the Eulerian velocity field $V(x,y,z) = (V_x, V_y, V_z)$ of a fluid motion has been known, we can use the equations

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z} = dt \tag{1}$$

to investigate the paths of the fluid particle motions and their properties. The Lagrangian de-

* **Received date**: Nov 24, 2003
Foundation item: State Key Laboratory Research Project(40023001), National Nature Scientific Foundation of China(40233029) and Foundation of Institute of Atmospheric Physics(8-0412)
Biography: DUAN Wan-suo, male, Han, Shanxi, associated professor, engages in climate predictability.

scription of the motion of (1) is a set of three-dimensional ordinary differential equations:

$$\frac{dx}{dt} = V_x(x,y,z), \frac{dy}{dt} = V_y(x,y,z), \frac{dz}{dt} = V_z(x,y,z). \tag{2}$$

In this system, "phase space" is referred as "configuration space". And the orbit of individual particle can be marked and followed. In Zaslavsky et al. [11], the authors discussed the field oı velocities defined by the following expressions:

$$\begin{aligned} V_x &= -\frac{\partial \Psi}{\partial y} + \varepsilon \sin z, \\ V_y &= \frac{\partial \Psi}{\partial x} - \varepsilon \cos z, \\ V_z &= \Psi, \end{aligned} \tag{3}$$

where ε is a parameter, the function $\Psi = \Psi(x,y)$ satisfies the two-dimensional Helmho-Itz equation:

$$\nabla^2 \Psi + \Psi = 0. \tag{4}$$

The flows of $(3)_\varepsilon$ satisfy the incompressibility condition $\mathrm{div}V = 0$ and the Beltrami condition $\mathrm{rot}V = -V$, which are an appropriate generalization of the ABC flows. If choosing the following expression: $\Psi = \psi_0 \sum_{j=1}^{q} \cos(Re_j)$, where $R(x,y)$ is the coordinate vector, and $e_j = \left\{\cos \frac{2\pi}{q} j, \sin \frac{2\pi}{q} j\right\}$ are unit vectors that shape a regular q- star, the flows of $(3)_\varepsilon$ are of the symmetry of the qth order, or quasi-symmetry. When $q = 3$ or 6, Zaslavsky et al. [11] has provided the evidence for the existence of chaos in the systems $(3)_\varepsilon$ with $\varepsilon \neq 0$ through numerical experiments. Furthermore, if the chaotic and resonant behavior of the symstem can be understood theoretically, it will be more significant. However, to the best of our knowledge, no paper has paid attention to this problem of $(3)_\varepsilon$.

In this paper, our aim is at the investigation of the analytical criteria for the existence of chaotic and resonant stream lines of the above quasi-symmetry flows $(3)_\varepsilon$. By using the Melnikov method[6,10], we obtain the conditions under which there exist the chaotic and resonant solutions of the system $(3)_\varepsilon$ in the sense of an generalization of the planar Smale-Birkhoff homoclinic theorem[3].

2. The Analysis of Unperturbed System

The qth ($q = 3$ or 6) quasi-symmetry flow with the field of velocities (3) is as the following dynamical system[11]:

$$\begin{aligned} \dot{x} &= \sqrt{3}\cos\frac{x}{2}\sin\frac{\sqrt{3}}{2}y + \varepsilon a \sin z, \\ \dot{y} &= -\sin x - \sin\frac{x}{2}\cos\frac{\sqrt{3}}{2}y - \varepsilon b \cos z, \\ \dot{z} &= \cos x + 2\cos\frac{x}{2}\cos\frac{\sqrt{3}}{2}y, \end{aligned} \tag{5}$$

where a and b are perturbed parameters, $0 < \varepsilon \ll 1$. By using transformation

$$(x, \sqrt{3}y, \sqrt{3}z, \sqrt{3}t) \rightarrow (u, v, w, \tau),$$

the system $(5)_\varepsilon$ can be rewritten into

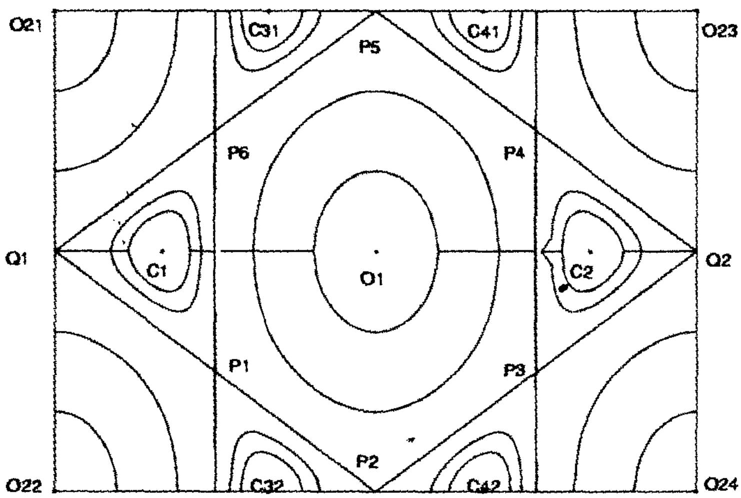
$$\begin{aligned}\dot{u} &= \cos\frac{u}{2}\sin\frac{v}{2} + \varepsilon a \sin\frac{\sqrt{3}}{3}w, \\ \dot{v} &= -\left(\sin u + \sin\frac{u}{2}\cos\frac{v}{2}\right) - \varepsilon b \cos\frac{\sqrt{3}}{3}w, \\ \dot{w} &= \cos u + 2\cos\frac{u}{2}\cos\frac{v}{2}.\end{aligned} \tag{6}$$

Clearly, when $\varepsilon = 0$, the first and second equation of $(6)_\varepsilon$ form a Hamiltonian system

$$\begin{aligned}\dot{u} &= \cos\frac{u}{2}\sin\frac{v}{2}, \\ \dot{v} &= -\left(\sin u + \sin\frac{u}{2}\cos\frac{v}{2}\right),\end{aligned} \tag{7}$$

which has the first integral

$$H(u,v) = \cos u + 2\cos\frac{u}{2}\cos\frac{v}{2} = h. \tag{8}$$



The system (7) is 4π - periodic in u and v respectively, so the state variables (u,v) can be regarded belonging to a two-dimensional torus $T^2 = S_1 \times S_1$, where $S_1 = [-2\pi, 2\pi]$ with -2π and 2π identified.

The flows defined by (7) is plotted in Fig. 1. On the two-torus T^2, there exist six-centers: $O_1(0,0)$, $C_1(-4/3\pi,0)$, $C_2(4/3\pi,0)$, C_3 ($(-2/3\pi,-2\pi)$ and $(-2/3\pi,2\pi)$ are identified), C_4 ($(2/3\pi,-2\pi)$ and $(2/3\pi,2\pi)$ are identified) and O_2 ($O_{21}(-2\pi,2\pi)$, $O_{22}(-2\pi,-2\pi)$, $O_{23}(2\pi,2\pi)$ and $O_{24}(-2\pi,2\pi)$ are identified). There exist six saddle points: $P_1(-\pi,-\pi)$, $P_2(0,-2\pi)$ and $P_5(0,2\pi)$ are identified, $P_3(\pi,-\pi)$, $P_4(\pi,\pi)$, $P_6(-\pi,\pi)$, $Q_1(-2\pi,0)$ and $Q_2(2\pi,0)$ are identified. It is easily demonstrated that the straightlines $u=-\pi$, $u=\pi$, $v=2\pi+u$, $v=-2\pi-u$, $v=-2\pi+u$ and $v=2\pi-u$ are invariant manifolds, which form heteroclinic orbits of (7). These straightlines partition T^2 into six period annuluses, which are full of families of periodic orbits of (7). For the above equilibria, the corresponding Hamiltonian quantities defined by (8) can be calculated:

$$H(O_1) = H(O_2) = 3, H(C_1) = H(C_2) = -3/2,$$
$$H(Q_1) = H(Q_2) = -3/2, H(P_i) = -1, i = 1,2,\cdots,6.$$

Consequently, depending on different Hamiltonian values h, the parametric representations of periodic families and heteroclinic orbits can be obtained:

(I) The family $\{\Gamma_1^h\}$ of closed orbits surrounding $O_1, h \in (-1,3)$:

$$u_1(\tau) = 2\arctan b_1 \operatorname{cn}(\Omega_1\tau, k_1),$$
$$v_1(\tau) = -2\arcsin \frac{2\Omega_1 b_1 \operatorname{sn}(\Omega_1\tau, k_1)\operatorname{dn}(\Omega_1\tau, k_1)}{\sqrt{1 + b_1^2 \operatorname{cn}^2(\Omega_1\tau, k_1)}},$$

where

$$\Omega_1^2 = \frac{\sqrt{3+2h}}{4},$$
$$k_1^2 = \frac{2\sqrt{3+2h} + 3 - h^2}{4\sqrt{3+2h}},$$
$$b_1 = \frac{\sqrt{2\sqrt{3+2h} - 2h}}{\sqrt{3+2h} - 1},$$

and $k_1 \to 1$ as $h \to -1$; $k \to 0$ as $h \to 3$.

(II) The family $\{\Gamma_2^h\}$ of closed orbits surrounding $O_2, h \in (-1,3)$:

$$u_2(\tau) = 2\pi + 2\arctan b_1 \operatorname{cn}(\Omega_1\tau, k_1),$$
$$v_2(\tau) = 2\pi - 2\arcsin \frac{2\Omega_1 b_1 \operatorname{sn}(\Omega_1\tau, k_1)\operatorname{dn}(\Omega_1\tau, k_1)}{\sqrt{1 + b_1^2 \operatorname{cn}^2(\Omega_1\tau, k_1)}}.$$

Clearly, the periodic orbits $\{\Gamma_1^h\}$ and $\{\Gamma_2^h\}$ have the period $T_1(k_1) = \frac{4K(k_1)}{\Omega_1}$ for a fixed $k_1 = k_1(h)$, where $K(k_1)$ is the complete elliptic integral of the first kind. When h increases from -1 to 3, T_1 decreases monotonously.

(III) The families $\{\Gamma_3^h\}$ and $\{\Gamma_4^h\}$ of closed orbits surrounding C_1 and C_2, $h \in (-3/2, -1)$:

$$u_{3,4}(\tau) = \pm 2\left(\pi + \arctan \frac{b^2}{\operatorname{dn}(\Omega_2\tau, k_2)}\right),$$
$$v_{3,4}(\tau) = \pm 2\arcsin \frac{\Omega_2 k_2^2 \operatorname{sn}(\Omega_2\tau, k_2)\operatorname{cn}(\Omega_2\tau, k_2)\operatorname{dn}(\Omega_2\tau, k_2)}{\sqrt{b_2^2 + \operatorname{dn}^2(\Omega_2\tau, k_2)}},$$

where

$$\Omega_2^2 = \frac{3 - h^2 + 2\sqrt{3+2h}}{16},$$
$$k_2^2 = \frac{4\sqrt{3+2h}}{3 - h^2 + 2\sqrt{3+2h}},$$
$$b_2 = -\frac{\sqrt{-2h - 2\sqrt{3+2h}}}{\sqrt{3+2h} + 1},$$

and $k_2 \to 0$ as $h \to -3/2$, $k_2 \to 1$ as $h \to -1$.

(IV) The families $\{\Gamma_5^h\}$ and $\{\Gamma_6^h\}$ of closed orbits surrounding C_3 and C_4, $h \in (-3/2, -1)$:

$$u_{5,6}(\tau) = \mp 2\arctan \frac{b_2}{\operatorname{dn}(\Omega_2\tau, k_2)},$$

$$v_{5,6}(\tau)=2\pi\pm 2\arcsin\frac{\Omega_2 k_2^2\mathrm{sn}(\Omega_2\tau,k_2)\mathrm{cn}(\Omega_2\tau,k_2)\mathrm{dn}(\Omega_2\tau,k_2)}{\sqrt{b_2^2+\mathrm{dn}^2(\Omega_2\tau,k_2)}}.$$

The periodic orbits $\{\Gamma_3^h\}-\{\Gamma_6^h\}$ have the same period $T_2(k_2)=\dfrac{2K(k_2)}{\Omega_2}$. It is easily shown that when h increases from $-3/2$ to -1, T_2 increases monotonously.

(V) The six heteroclinic orbits surrounding O_1, $h=-1$:

$$P_1P_2:\begin{cases}u(\tau)=-2\arctan(e^{\tau/2}),\\ v(\tau)=-2\pi+2\arctan(e^{\tau/2}),\end{cases}\quad P_2P_3:\begin{cases}u(\tau)=2\arctan(e^{-\tau/2}),\\ v(\tau)=-2\pi+2\arctan(e^{-\tau/2}),\end{cases}$$

$$P_3P_4:\begin{cases}u(\tau)=\pi,\\ v(\tau)=-\pi+4\arctan(e^{-\tau/2}),\end{cases}\quad P_4P_5:\begin{cases}u(\tau)=2\arctan(e^{\tau/2}),\\ v(\tau)=-2\pi-2\arctan(e^{\tau/2}),\end{cases}$$

$$P_5P_6:\begin{cases}u(\tau)=-2\arctan(e^{-\tau/2}),\\ v(\tau)=-2\pi-2\arctan(e^{-\tau/2}),\end{cases}\quad P_6P_1:\begin{cases}u(\tau)=-\pi,\\ v(\tau)=-\pi+4\arctan(e^{\tau/2}).\end{cases}$$

(VI) The six heteroclinic orbits surrounding C_1 and C_2 respectively

$$P_6Q_1:\begin{cases}u(\tau)=-\pi-2\arctan(e^{\tau/2}),\\ v(\tau)=\pi-2\arctan(e^{\tau/2}),\end{cases}\quad Q_1P_1:\begin{cases}u(\tau)=-\pi-2\arctan(e^{-\tau/2}),\\ v(\tau)=-\pi+2\arctan(e^{-\tau/2}),\end{cases}$$

$$P_1P_6:\begin{cases}u(\tau)=-\pi,\\ v(\tau)=-\pi+4\arctan(e^{\tau/2}),\end{cases}\quad P_4Q_2:\begin{cases}u(\tau)=\pi+2\arctan(e^{-\tau/2}),\\ v(\tau)=\pi-2\arctan(e^{-\tau/2}),\end{cases}$$

$$Q_2P_3:\begin{cases}u(\tau)=\pi+2\arctan(e^{\tau/2}),\\ v(\tau)=-\pi+2\arctan(e^{\tau/2}),\end{cases}\quad P_3P_4:\begin{cases}u(\tau)=\pi,\\ v(\tau)=-\pi+4\arctan(e^{-\tau/2}).\end{cases}$$

3. The Resonant Stream Lines of Perturbed System

In this section, we investigate the existence of the resonant stream lines of $(6)_\varepsilon$ by employing Melnikov function. Firstly rewrite the system $(6)_\varepsilon$ into the following form:

$$u'=\frac{\cos\frac{u}{2}\sin\frac{v}{2}}{\cos u+2\cos\frac{u}{2}\cos\frac{v}{2}}+\varepsilon a\frac{\sin\frac{\sqrt{3}}{3}w}{\cos u+2\cos\frac{u}{2}\cos\frac{v}{2}},$$

$$v'=\frac{-\left(\sin u+\sin\frac{u}{2}\cos\frac{v}{2}\right)}{\cos u+2\cos\frac{u}{2}\cos\frac{v}{2}}-\varepsilon b\frac{\cos\frac{\sqrt{3}}{3}w}{\cos u+2\cos\frac{u}{2}\cos\frac{v}{2}}, \tag{9}$$

where "$'$" is the derivative with respect to w. It is easily derived that the system $(9)_{\varepsilon=0}$ is a integrable Hamiltonian system and has a first integral (8), i. e. $\cos u+2\cos\frac{u}{2}\cos\frac{v}{2}=h$.

From the results obtained in section 2, it is clear that the system $(9)_{\varepsilon=0}$ has the same equilibria as the ones of (7), and their phase-portraits are also common except that "w" is the new "time" variable, where, in fact, w satisfies the equation $\dfrac{\mathrm{d}w}{\mathrm{d}\tau}=h$. Hence, if taking the initial value as w_0, then $w=w_0+h\tau$. To facilitate the following discussions, without loss of generality, we suppose $w_0=0$.

Based on all above these analysis, the parametric expressions of orbits of $(9)_{\varepsilon=0}$ can be easily obtained by substituting $\lambda_1 w$ for $\Omega_1\tau$ when $-1<h<3$ and $\lambda_2 w$ for $\Omega_2\tau$ when $-3/2<$

$h < -1$ respectively, where $\lambda_i = \frac{\Omega_i}{|h|}(i = 1, 2)$, and $-w$ for τ when $h = -1$.

Now we compute the subharmonic Melnikov function[6,10] of the periodic orbits in $\{\Gamma_1^h\}$, $-1 < h < 3$, by using the resonant condition

$$T_1(h) = \frac{4|h|K(k_1)}{\Omega_1} = 2\sqrt{3}m\pi. \tag{10}$$

For every $k_1 = k_1(m, 1)$ defined by (10), using the formula of Langebartel[7], the corresponding subharmonic Melnikov function along the periodic orbit Γ_1^h (where $h = h(k_1)$) is derived as follows:

$$\begin{aligned} h^2 M_1(w) &= \int_0^{T_1(h)} \left[a\left(\sin u(\tau) + \sin\frac{u(\tau)}{2}\cos\frac{v(\tau)}{2}\right)\sin\frac{\sqrt{3}}{3}(\tau + w) \right. \\ &\qquad \left. - b\cos\frac{u(\tau)}{2}\sin\frac{v(\tau)}{2}\cos\frac{\sqrt{3}}{3}(\tau + w)\right]d\tau \\ &= (J_{11}(m,1) + J_{12}(m,1) + J_{13}(m,1))\sin\frac{\sqrt{3}}{3}w, \end{aligned} \tag{11}$$

where

$$\begin{aligned} J_{11}(m,1) &= a\int_0^{T_1(h)} \sin u(\tau)\cos\frac{\sqrt{3}}{3}\tau d\tau \\ &= \frac{ab_1|h|}{\Omega_1(1 + b_1^2)}\int_{-2K}^{2K}\left[\frac{\mathrm{cn}(s,k_1)}{1 - \beta_1\mathrm{sn}(s,k_1)} + \frac{\mathrm{cn}(s,k_1)}{1 + \beta_1\mathrm{sn}(s,k_1)}\right]\cos\frac{m\pi s}{2K}ds \\ &= \frac{2ab_1\pi}{\lambda_1(1 + b_1^2)\sqrt{k_1^2 - \beta_1^2}}\sin\frac{s_0}{\sqrt{3}\lambda_1}\mathrm{sech}\frac{K'}{\sqrt{3}\lambda_1}\sin\frac{m\pi}{2}, \quad m \text{ is odd}; \\ J_{13}(m,1) &= -b\int_0^{T_1(h)} \cos u(\tau)\sin\frac{v(\tau)}{2}\sin\frac{\sqrt{3}}{3}\tau d\tau \\ &= \frac{-bb_1|h|}{1 + b_1^2}\int_{-2K}^{2K}\left[\frac{\mathrm{dn}(s,k_1)}{1 - \beta_1\mathrm{sn}(s,k_1)} + \frac{\mathrm{dn}(s,k_1)}{1 + \beta_1\mathrm{sn}(s,k_1)}\right]\sin\frac{m\pi s}{2K}ds \\ &= \frac{-2b\Omega_1\pi}{\lambda_1\sqrt{(1 + b_1^2)(1 - \beta_1^2)}}\cos\frac{s_0}{\sqrt{3}\lambda_1}\mathrm{sech}\frac{K'}{\sqrt{3}\lambda_1}\sin\frac{m\pi}{2}, \quad m \text{ is odd}, \end{aligned}$$

and

$$\begin{aligned} J_{12}(m,1) &= a\int_0^{T_1(h)} \sin\frac{u(\tau)}{2}\cos\frac{v(\tau)}{2}\cos\frac{\sqrt{3}}{3}\tau d\tau \\ &= \frac{ab_1}{\lambda_1}\int_{-2K}^{2K}\frac{\mathrm{cn}(s,k_1)\sqrt{1 + b_1^2\mathrm{cn}^2(s,k_1) - 4\Omega_1^2 b_1^2\mathrm{sn}^2(s,k_1)\mathrm{dn}^2(s,k_1)}}{1 + b_1^2\mathrm{cn}^2(s,k_1)}\cos\frac{m\pi s}{2K}ds, m \text{ is odd}, \end{aligned}$$

where $\beta_1^2 = \frac{b_1^2}{1 + b_1^2}$, $\mathrm{dn}(s_0, k_1) = \sqrt{\frac{h^2 - 3 + 2\sqrt{3 + 2h}}{2(2 + h)\sqrt{3 + 2h} - 6 - 4h}}$, $0 < s_0 < K(k_1)$, $K'(k_1) = K(k'_1)$, $k'_1 = \sqrt{1 - k_1^2}$, h and k_1 satisfy (10).

From the above discussions, it is easily shown that the condition

$$J_{11}^2(m,1) + J_{12}^2(m,1) + J_{13}^2(m,1) \neq 0 \tag{12}$$

holds. The function $M_1(w)$ has simple zeros. Based on the generalization of planar Smale-Birkhoff homoclinic theorem to the case of a heteroclinic saddles connection containing a finite number of fixed points[3], it is concluded that the periodic orbits Γ_1^h and Γ_2^h of the unperturbed system bifurcate odd number order resonant orbits in perturbed system.

For the resonant periodic orbits $\{\Gamma_3^h\}$, $-3/2 < h < -1$, the resonant condition is

$$T_2(h) = \frac{2 \mid h \mid K(k_2)}{\Omega_2} = 2\sqrt{3}m\pi. \tag{13}$$

Similarly their subharmonic Melnikov function can be computed easily. For every $k_2 = k_2(m, 1)$ defined by (13) and the periodic orbits $\{\Gamma_3^h\}$. (where $h = h(k_2)$), the corresponding subharmonic Melnikov is

$$\begin{aligned} h^2 M_3(w) &= \int_0^{T_2(h)} \Big[a\Big(\sin u(\tau) + \sin\frac{u(\tau)}{2}\cos\frac{v(\tau)}{2}\Big)\sin\frac{\sqrt{3}}{3}(\tau + w) \\ &\quad - b\cos\frac{u(\tau)}{2}\sin\frac{v(\tau)}{2}\cos\frac{\sqrt{3}}{3}(\tau + w)\Big]d\tau \\ &= (J_{31}(m,1) + J_{32}(m,1) + J_{33}(m,1))\sin\frac{\sqrt{3}}{3}w, \end{aligned} \tag{14}$$

where

$$\begin{aligned} J_{31}(m,1) &= a\int_0^{T_2(h)} \sin u(\tau)\cos\frac{\sqrt{3}}{3}\tau d\tau \\ &= \frac{ab_2}{\lambda_2(1+b_2^2)}\int_{-K}^{K}\Big[\frac{\mathrm{dn}(s,k_2)}{1-\beta_2\mathrm{sn}(s,k_2)} + \frac{\mathrm{dn}(s,k_2)}{1+\beta_2\mathrm{sn}(s,k_2)}\Big]\cos\frac{m\pi s}{K}ds \\ &= \frac{ab_2\pi}{\lambda_2(1+b_2^2)\sqrt{1-\beta_2^2}}\cos\frac{s_0}{\sqrt{3}\lambda_2}\mathrm{sech}\frac{K'}{\sqrt{3}\lambda_2}\cos\frac{m\pi}{2}, \quad m \text{ is even}, \end{aligned}$$

$$\begin{aligned} J_{32}(m,1) &= a\int_0^{T_2(h)} \sin\frac{u(\tau)}{2}\cos\frac{v(\tau)}{2}\cos\frac{\sqrt{3}}{3}\tau d\tau \\ &= -\frac{ab_2}{\lambda_2}\int_{-K}^{K}\frac{\sqrt{b_2^2 + \mathrm{dn}^2(s,k_2) - \Omega_2^2 k_2^4 \mathrm{sn}^2(s,k_2)\mathrm{cn}^2(s,k^2\mathrm{dn}^2)(s,k_2)}}{b_2^2 + \mathrm{dn}^2(s,k_2)}\cos\frac{\pi m s}{K}ds, \\ &\qquad m \text{ is even}, \end{aligned}$$

and

$$\begin{aligned} J_{33}(m,1) &= -b\int_0^{T_2(h)} \cos\frac{u(\tau)}{2}\sin\frac{v(\tau)}{2}\cos\frac{\sqrt{3}}{3}\tau d\tau \\ &= \frac{b\Omega_2 k_2^2}{\lambda_2}\int_{-K}^{K}\frac{\mathrm{sn}(s,k_2)\mathrm{cn}(s,k_2)\mathrm{dn}^2(s,k_2)}{b_2^2 + \mathrm{dn}^2}\cos\frac{m\pi s}{K}ds, \quad m \text{ is even}, \end{aligned}$$

where $\beta_2^2 = \frac{k_2^2}{1+b_2^2}$, $\mathrm{dn}(s_0, k_2) = \sqrt{\frac{3-h^2-2\sqrt{3+2h}}{3-h^2}}$, $0 < s_0 < K(k_2)$, $K'(k_2) = K(k'_2)$, $k'_2 = \sqrt{1-k_2^2}$, h and k_2 satisfy (13). Similar to the case of $\{\Gamma_1^h\}$ the condition

$$J_{31}^2(m,1) + J_{32}^2(m,1) + J_{33}^2(m,1) \neq 0 \tag{15}$$

holds. The function $M_3(w)$ has simple zeros, which indicates that the periodic orbits $\Gamma_3^h - \Gamma_6^h$ bifurcate even number order resonant stream lines in the perturbed system.

The above results can be outlined as follows:

Theorem 1 For the perturbed system $(5)_\varepsilon$, $0 < \varepsilon \ll 1$, the resonant orbits Γ_1^h and Γ_2^h of the corresponding unperturbed system bifurcate old number order periodic resonant stream lines, and the resonant orbits $\Gamma_3^h - \Gamma_6^h$ bifurcate even number order periodic resonant stream lines.

4. The Chaotic Stream Lines of Perturbed Systems

By using the parametric expression given by section 2 and 3, the Melnikov function along the heteroclinic orbits can be obtained:

$$M(w) = \int_{-\infty}^{\infty} \frac{a\left(\sin u(\tau) + \sin\frac{u(\tau)}{2}\cos\frac{v(\tau)}{2}\right)\sin\frac{\sqrt{3}}{3}(\tau + w) - b\cos\frac{u(\tau)}{2}\sin\frac{u(\tau)}{2}\cos\frac{\sqrt{3}}{3}(\tau + w)}{\left(\cos u(\tau) + 2\cos\frac{u(\tau)}{2}\cos\frac{v(\tau)}{2}\right)^2} d\tau.$$

Since

$$\cos u(\tau) + 2\cos\frac{u(\tau)}{2}\cos\frac{v(\tau)}{2} = -1,$$

it can be reduced to

$$M(w) = \int_{-\infty}^{\infty}\left[a\left(\sin u(\tau) + \sin\frac{u(\tau)}{2}\cos\frac{v(\tau)}{2}\right)\sin\frac{\sqrt{3}}{3}(\tau + w) - b\cos\frac{u(\tau)}{2}\sin\frac{v(\tau)}{2}\cos\frac{\sqrt{3}}{3}(\tau + w)\right]d\tau.$$

Using the parametric expressions (V) and (VI) in section 2 and the formula

$$\int_{-\infty}^{\infty}\left[\cos\frac{\sqrt{3}}{3}\tau/\mathrm{ch}\,\frac{\tau}{2}\right]dt = 2\pi\,\mathrm{sech}\,\frac{\sqrt{3}}{3}\pi,$$

we obtain

$$M_{P_1P_2}(w) = M_{P_4Q_2}(w) = -M_{P_4P_5}(w) = -M_{Q_1P_1}(w) = \left(\frac{-\pi}{\sqrt{9a^2+b^2}}\mathrm{sech}\,\frac{\sqrt{3}}{3}\pi\right)\sin\left(\frac{\sqrt{3}}{3}w + \varphi_0\right),$$

$$M_{P_2P_3}(w) = M_{P_6Q_1}(w) = -M_{Q_2P_3}(w) = -M_{P_5P_6}(w) = \left(\frac{\pi}{\sqrt{9a^2+b^2}}\mathrm{sech}\,\frac{\sqrt{3}}{3}\pi\right)\sin\left(\frac{\sqrt{3}}{3}w - \varphi_0\right),$$

$$M_{P_3P_4}(w) = -M_{P_6P_1}(w) = 2a\pi\,\mathrm{sech}\,\frac{\sqrt{3}}{3}\pi\sin\frac{\sqrt{3}}{3}w,$$

where $\tan\varphi_0 = b/3a$. It is easily seen that each of these Melnikov functions has simple zeros when $ab \neq 0$. This indicates that there exist transverse heteroclinic six-cycles and transverse three-cycles for Poincare map of (9). It follows the existence of transverse homoclinic points of (9)(see [3]). So we have the following conclusion:

Theorem 2 For $0 < \varepsilon \ll 1$, if $ab \neq 0$, then the system $(5)_\varepsilon$ has chaotic stream lines in the sense of the existence of the Smale horseshoes.

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准对称流的混沌和共振流线

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摘要：采用计算 Melnikov 函数的方法，研究了描述 q^{th}（$q=3$ 或 6）准对称流流体粒子运动的动力系统. 文中在分析未扰动系统轨道解析表示的基础上，深入考察了扰动系统的分岔情况. 结果表明，扰动系统在一定条件下能够分支出混沌和共振流线.

关键词：共振流线；混沌流线；Smale 马蹄；异宿环

准对称流的混沌和共振流线

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刊名： 应用数学 ISTIC PKU
英文刊名： MATHEMATICA APPLICATA
年，卷(期)： 2004,17(4)

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