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APPLICATIONS OF NONLINEAR OPTIMIZATION METHOD TO NUMERICAL STUDIES OF ATMOSPHERIC AND OCEANIC SCIENCES *

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Abstract: Linear singular vector and linear singular value can only describe the evolution of sufficiently small perturbations during the period in which the tangent linear model is valid. With this in mind, the applications of nonlinear optimization methods to the atmospheric and oceanic sciences are introduced, which include nonlinear singular vector (NSV) and nonlinear singular value (NSVA), conditional nonlinear optimal perturbation (CNOP), and their applications to the studies of predictability in numerical weather and climate prediction. The results suggest that the nonlinear characteristics of the motions of atmosphere and oceans can be explored by NSV and CNOP. Also attentions are paid to the introduction of the classification of predictability problems, which are related to the maximum predictable time, the maximum prediction error, and the maximum allowing error of initial value and the parameters. All the information has the background of application to the evaluation of products of numerical weather and climate prediction. Furthermore the nonlinear optimization methods of the sensitivity analysis with numerical model are also introduced, which can give a quantitative assessment whether a numerical model is able to simulate the observations and find the initial field that yield the optimal simulation. Finally, the difficulties in the lack of ripe algorithms are also discussed, which leave future work to both computational mathematics and scientists in geophysics.

Key words: nonlinear optimization; weather; climate; predictability; sensitivity analysis

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Introduction

Numerical prediction of weather and climate essentially consists of solving a set of partial differential equations, which is usually referred as a "model" in our scientific literature, with proper initial and boundary values. Due to the great difficulties for the theoretical study posed by

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the complexity and nonlinearity of atmospheric and oceanic motions, numerical modeling has become a common approach in the world. Consequently, numerical model shows its superiority in scientific background for the task of numerical weather and climate prediction. But the inevitable model deficiencies and initial errors will cause the uncertainties of forecast results. The studies of these uncertainties have become known as "predictability problems", which can be traced back to the last century, some fundamental notions can be found in Refs. [1,2], and the latter Ref. [3]. Until now, the predictability problems of numerical weather and climate prediction are still one of the important subjects, *e.g.*, the well-known international research programme on "Climate Variability and Predictability (CLIVAR)".

Many authors employed linear singular vector (LSV) and linear singular value (LSVA) to investigate the predictability of atmospheric and oceanic motions[4,5]. Usually, it is assumed that the initial perturbation is sufficiently small such that its evolution can be governed by the tangent linear model (TLM)[6], which is derived by linearizing the corresponding nonlinear model about a basic solution. For a discrete TLM, the forward propagator can be expressed as a matrix, and computing the LSV is reduced to calculate the singular vector corresponding to the maximum singular value of the matrix. LSV and LSVA were proposed by Lorenz[6] to study the predictability of atmospheric motion. Then they were also utilized to study the finite-time linear stability[7] and to construct the initial perturbations field for the ensemble forecast at the European Center for Medium-Range Weather Forecast (ECMWF). Recently, this method has been used to find out the initial condition for optimal growth in a coupled ocean-atmosphere model of E1 Nino-Southern Oscillation (ENSO), in an attempt to explore error growth and the predictability of the coupled model[8,9].

The motions of atmosphere and ocean are nonlinear and complex and are usually governed by complicated nonlinear model. LSV and LSVA are established on the condition that the evolution of the initial perturbation can be described approximately by the linear version of the nonlinear model. This raises a few questions concerning the validity of TLM. Although there have been a few papers to address these concerns, their conclusions are accordant, that is, determining the validity of TLM in advance is difficult and essentially empirical[10-12]. Therefore, in order to investigate the effect of nonlinearity on the predictability of atmospheric and oceanic motions, based on the research for several decades, Mu and Duan[13,14] proposed the concepts of nonlinear singular vector (NSV) and conditional nonlinear optimal perturbation, which are discussed in detail in Section 1 and Section 2 respectively.

According to the factors causing the uncertainties of forecast results, the predictability problems are usually classified into two types, the first kind of predictability, which is related to the initial errors, and the second kind of predictability, which is to the model errors[15]. The definition of model errors varies with the different authors[16]. In this paper, we adopt the following one: If the initial value of the model is the true state, then the difference between the value of the forecast and the true state at the prediction time is called model error[17]. From this definition, it is easily seen that there are many factors causing model errors. However, in this paper, we only consider the errors of the parameters in the model, which is generally regarded as one of the main problems in the model[16]. On the basis of practical demands, Mu *et al*.[16] classified the three problems of predictability, which are referred to in Section 3.

Initial errors and model errors are the dominant factors causing the uncertainties of forecast results. In order to reduce and identify them from the model, the meteorologist performed numerous works with sensitivity analysis. Usually, there are three approaches utilized in sensitivity analysis: numerical simulation, adjoint method, and LSV[18-20]. For the approach of numerical simulation, it is often blindfold and empirical in the process of choosing the control experiments[21]. The adjoint method and LSV are based on the tangent linear model, which can only describe the evolution of small perturbations in the time period, in which the tangent linear model is valid[22]. In Section 4, the authors will introduce the preliminary applications of nonlinear optimization method to the sensitivity analysis of the numerical model.

1 Nonlinear Singular Vector and Nonlinear Singular Value

The model, which governs the motions of atmosphere and ocean, can be written as the following partial differential equations with initial and boundary values,

$$\begin{cases} \dfrac{\partial w}{\partial t} + F(w) = 0, \\ w\mid_{t=0} = w_0, \end{cases} \tag{1}$$

where $w(x,t) = (w_1(x,t), w_2(x,t), \cdots, w_n(x,t)), x = (x_1, x_2, \cdots, x_n)$, $(x,t) \in \Omega \times [0, T], T < +\infty$, t is the time, F a nonlinear operator, w_0 the initial state and Ω a region in R^n.

Throughout this paper, the initial value problem (1) is assumed to be well posed, which means that for each initial value w_0, there exists a unique solution of Eq.(1) which depends on w_0 continuously.

Let U_0 be the initial value of the basic state $U(x,t)$. If $u_0(x)$ is the initial perturbation superposed on U_0, $U_0 + u_0$ will evolve into $U(x,T) + u(x,T)$ at time T. Then $u(x,T)$ is the nonlinear evolution of initial perturbation $u_0(x)$.

The initial perturbation u_{10}^* is called the first nonlinear singular vector (NSV), or the nonlinear optimal perturbation superposed on the basic state $U(x,t)$, if and only if

$$I(u_{10}^*) = \max_{u_0} \frac{\| u(T) \|^2}{\| u_0 \|^2}. \tag{2}$$

The positive square root of $I(u_{10}^*)$ is the nonlinear singular value (NSVA), which represents the growth rate of the first NSV in terms of the chose norm $\| \cdot \|$. Here it is worthwhile to point out that there could exist several first NSVs corresponding to the first NSVA[13].

In addition to the first NSV defined above, we can also define the second NSV, u_{20}^*, by investigating the following conditional maximum problem:

$$I(u_{20}^*) = \max_{u_0 \perp u_{10}^*} \frac{\| u(T) \|^2}{\| u_0 \|^2}, \tag{3}$$

where $u_0 \perp u_{10}^*$ means u_0 is orthogonal to all first NSV u_{10}^*, the positive square root of $I(u_{20}^*)$ is called the second nonlinear singular value. Obviously, the nth$(n = 3,4,\cdots,n)$ NSV and singular value can be defined in this manner step by step.

The two-dimensional quasi-geotropic model was used to study the NSV and NSVA[23]. The numerical results demonstrated that if the first NSV in terms of the chosen norm was sufficiently small, it could be approximated by the LSV. But for the large nonlinear optimal perturbation, the tangent linear model could not be used to describe its nonlinear evolution. Besides some types of basic states, there existed local nonlinear optimal perturbations, which corresponded to the local maximum values of the functional $I(\boldsymbol{u}_0)$. However there was no such phenomenon in the case of LSVs and LSVAs due to the absence of the corresponding TLM. These local nonlinear optimal perturbations usually possess larger norms compared with the first NSV, which corresponds to the global maximum value of functional $I(\boldsymbol{u}_0)$. Although the growth rate of the local nonlinear optimal perturbations are smaller than the first NSVA, their nonlinear evolutions at the end of the time interval are considerably greater than that of the first NSV in terms of the chosen norm. In this case, the local nonlinear optimal perturbation could play a more important role than the global nonlinear optimal perturbation in the study of the predictability.

Recently Durbiano[24] successfully computed the first six NSVs of a shallow water model, and compared them with the LSV to explore the difference between them.

It is clear from Refs. [23,24] that for the predictability, we should find out all the local nonlinear optimal perturbations, then investigate their effects on the predictability. But this is inconvenient in practical applications. Besides, the local nonlinear optimal perturbations with large norm could be insignificant physically.

All these weaknesses suggest that we should investigate the nonlinear optimal perturbation with constraint conditions.

2 Conditional Nonlinear Optimal Perturbation

Let $\boldsymbol{U}(\boldsymbol{x},t)$ and $\boldsymbol{U}(\boldsymbol{x},t)+\boldsymbol{u}(\boldsymbol{x},t)$ be the solutions of Eq. (1), of which the initial values are respectively $\boldsymbol{U}_0$ and $\boldsymbol{U}_0+\boldsymbol{u}_0$, $\boldsymbol{u}_0$ the initial perturbation on $\boldsymbol{U}(\boldsymbol{x},t)$. For the chosen norm $\|\cdot\|$, the initial perturbation $\boldsymbol{u}_{0\delta}$ is conditional nonlinear optimal perturbation (CNOP)[26], if and only if

$$J(\boldsymbol{u}_{0\delta}) = \max_{\|\boldsymbol{u}_0\| \leqslant \delta} \| M_T(\boldsymbol{U}_0+\boldsymbol{u}_0) - M_T(\boldsymbol{U}_0) \|, \tag{4}$$

where M_T is the numerical model, or the propagator from 0 to T, the inequality $\|\boldsymbol{u}_0\| \leqslant \delta$ is the constraint condition. It is easily shown from definition of CNOP that the nonlinear evolution of CNOP is the maximum among all the initial perturbations in $\|\boldsymbol{u}_0\| \leqslant \delta$.

In the above, the constraint condition is simply expressed as a ball with the given norm $\|\cdot\|$. Obviously, we can also investigate the situation that the initial perturbation belongs to some functional set, or satisfy some physical laws.

2.1 Applications of CNOP to predictability of ENSO

El Nino (La Nina) is a phenomenon of short-term climate variation happening in the tropical Pacific Ocean. Usually the annual mean sea surface temperature (SST) over the equatorial Pacific takes on a strong asymmetry between the relatively warm western part of the basin, the region is called the warm pool, and the cooler eastern basin, called the cold tongue. In some years, the SST anomaly of the equatorial eastern Pacific is up to a few degrees, the phenomenon of which is usually as El Nino (La Nina). This phenomenon is associated with the atmosphere, and thus the term ENSO (El Nino-Southern oscillation) that incorporates the southern oscillation phenomenon is commonly used[25]. Southern oscillation refers to a seesaw shift in surface air pressure at Darwin, Australia and the South Pacific Island of Tahiti. When the pressure is high at Darwin it is low at Tahiti and vice versa. Though they originate in the tropical Pacific, they have an impact on weather and climate globally. ENSO has to be regarded as an inherently coupled atmosphere-ocean mode.

Conditional nonlinear optimal perturbation (CNOP) was used to study the optimal precursors of ENSO event within the frame of a simple coupled ocean-atmosphere model for ENSO[26]. The results suggested that for the proper constraint condition, the CNOPs of the climatological mean state evolved into ENSO events more probably than the LSV. Consequently it was reasonable to regard CNOP as the optimal precursors of ENSO events. Observed anomalous monthly mean SST (°C) and depth of 20 °C isotherm (m) derived from NCEP ocean reanalysis for the equatorial eastern Pacific (5°S-5°N, 150°-90°W) region verified the existence of these optimal precursors qualitatively.

The "spring predictability barrier" problem for ENSO event, which is one of the essential characteristics of ENSO and means that regardless of when a forecast is started, at the time of the year the anomaly correlation falls rapidly, was also investigated. In Ref. [14], the CNOPs of El Nino and La Nina events were computed. The results suggested that the error growth was enhanced in spring for El Nino event and suppressed in spring for La Nina event. That was to say, there was a tendency of spring predictability barrier for El Nino and not for La Nina. And the larger the initial perturbation was, the more notable the phenomenon of spring predictability barrier was. There was also evidence in the paper of Mu and Duan[14] that LSV could only describe the evolution of the sufficiently small perturbation at the time interval when the tangent linear model was valid. For the large initial perturbation, LSV could not disclose that the effect of nonlinearity on the spring predictability barrier. Besides, LSV largely underestimated or overestimated the error growth of ENSO.

To further investigate what causes the spring predictability barrier in the model, the CNOPs of El Nino and La Nina events with strong and weak-coupled ocean-atmosphere instability were also computed[14]. It was shown that the strong-coupled ocean-atmosphere instability during spring of the year was one of the causes of the spring predictability barrier. Some sensitivity experiments showed that the spring predictability barrier of ENSO event had the tendency for phase-locking to the annual cycle of the climate mean state.

2.2 Applications of CNOP to sensitivity analysis of thermohaline circulation

The sensitivity problem of the ocean thermohaline circulation has been one of the fundamental researches in climate variability. Due to the presence of several physically nonlinear feedbacks processes that govern the evolution of the thermohaline flow, the thermohaline circulation shows the strong sensitivity to the finite perturbations, which largely limits the predictability of the thermohaline flow. Many authors employed the LSV to study the sensitivity of the thermohaline to the initial perturbation[27]. But the results obtained by LSV could not explain the asymmetry which responds to the sign of fresh water perturbation.

In order to explore the effect of nonlinearity on the sensitivity of the thermohaline, Mu *et al.*[28] adopted CNOP approach to analyze the different sensitivities of the thermohaline circulation to finite amplitude freshwater and salt perturbations. Within the frame of a simple model for the thermohaline circulation, the impacts of nonlinearity on the evolution of the finite amplitude freshwater and salinity perturbations were investigated. It was demonstrated that there existed an asymmetric nonlinear response to the sign of the finite fresh water perturbations. From the sensitivity analysis of the thermohaline circulation to the freshwater and salinity perturbations along the bifurcation diagram, it was shown that the system became unstable near the bifurcation diagram regime, and a finite perturbation could lead to a transition of the thermohaline circulation from an equilibrium state to another one.

3 Three Sub-Problems of Predictability

With the development of the human society and economy, people require to know the answers to the questions such as how large the prediction error is, and with given accuracy how long we can predict the weather or climate. With this in mind, Mu *et al*.[16] classified three predictability problems in numerical weather and climate prediction according to practical demands.

Problem 1 Suppose that the initial observation $\boldsymbol{u}_0^{\mathrm{obs}}$ and the first given value of the parameter $\boldsymbol{\mu}^{\mathrm{g}}$ are known, M_t and M_T are respectively the propagators from 0 to t and T. At prediction time T, the maximum allowing prediction error in terms of the chosen norm $\|\cdot\|_A$ is

$$\| M_T(\boldsymbol{u}_0^{\mathrm{obs}}, \boldsymbol{\mu}^{\mathrm{g}}) - \boldsymbol{u}_T^{\mathrm{t}} \|_A \leqslant \varepsilon, \tag{5}$$

where $\boldsymbol{u}_T^{\mathrm{t}}$ is the true value of the state at time T. Then the maximum predictable time, T_ε, can be estimated by the following nonlinear optimization problem:

$$T_\varepsilon = \max\{\tau \mid \| M_t(\boldsymbol{u}_0^{\mathrm{obs}}, \boldsymbol{\mu}^{\mathrm{g}}) - \boldsymbol{u}_t^{\mathrm{t}} \|_A \leqslant \varepsilon, 0 \leqslant t \leqslant \tau\}. \tag{6}$$

Since the true state $\boldsymbol{u}_t^{\mathrm{t}}$ cannot be obtained exactly, it is impossible to obtain the exact value of T_ε. However, if we know more information about the errors of initial value and the parameters, useful estimation on T_ε can be derived by using some methods. For example, assume that the errors of the initial value and the first given value of the parameters are known as follows:

$$\| \boldsymbol{u}_0^{\mathrm{t}} - \boldsymbol{u}_0^{\mathrm{obs}} \|_A \leqslant \delta_1, \quad \| \boldsymbol{\mu}^{\mathrm{t}} - \boldsymbol{\mu}^{\mathrm{g}} \|_B \leqslant \delta_2, \tag{7}$$

where $\|\cdot\|_B$ is a norm measuring the parameters in the model. Then we can investigate the following nonlinear optimization problem:

$$T_{\varepsilon l} = \min_{\boldsymbol{u}_0 \in B_{\delta_1}, \boldsymbol{\mu} \in B_{\delta_2}} \Big\{ T_{\boldsymbol{u}_0,\boldsymbol{\mu}} \mid T_{\boldsymbol{u}_0,\boldsymbol{\mu}} = \max \tau, \\ \| M_t(\boldsymbol{u}_0, \boldsymbol{\mu}) - M_t(\boldsymbol{u}_0^{\mathrm{obs}}, \boldsymbol{\mu}^{\mathrm{g}}) \| \leqslant \varepsilon, 0 \leqslant t \leqslant \tau \Big\}, \tag{8}$$

where B_{δ_1} and B_{δ_2} are the balls with centers at $\boldsymbol{u}_0^{\mathrm{obs}}$, $\boldsymbol{\mu}^{\mathrm{g}}$, and radius δ_1, δ_2, respectively.

It is not difficult to prove that

$$T_{\varepsilon l} \leqslant T_\varepsilon.$$

Thus $T_{\varepsilon l}$ of Eq. (8) gives the lower bound of the maximum predictable time.

Problem 2 Assume that the initial observation $\boldsymbol{u}_0^{\mathrm{obs}}$ and the first value of the parameter $\boldsymbol{\mu}^{\mathrm{g}}$ are given. At prediction time T, the prediction error can be expressed as follows:

$$E = \| M_T(\boldsymbol{u}_0^{\mathrm{obs}}, \boldsymbol{\mu}^{\mathrm{g}}) - \boldsymbol{u}_t^{\mathrm{t}} \|_A. \tag{9}$$

The true state $\boldsymbol{u}_T^{\mathrm{t}}$ cannot be obtained exactly, so E is not solvable. But if Eq. (7) holds, the prediction error E can be estimated by the following nonlinear optimization problem

$$E_u = \max_{\boldsymbol{u}_0 \in B_{\delta_1}, \boldsymbol{\mu} \in B_{\delta_1}} \| M_T(\boldsymbol{u}_0, \boldsymbol{\mu}) - M_T(\boldsymbol{u}_0^{\mathrm{obs}}, \boldsymbol{\mu}^{\mathrm{g}}) \|_A. \tag{10}$$

It is easy to prove that $E \leqslant E_u$. Then E_u gives the estimation of the upper bound of prediction error.

Problem 3 Assume that the initial observation u_0^{obs} and the first given value of the parameter μ^g are available. At prediction time T, the allowing maximum prediction error is Eq.(5). Then the allowing maximum initial error and the parameter error can be reduced into the nonlinear optimization problem

$$\delta_{max} = \max_{\delta}\{\delta \mid \| u_0 - u_0^{obs} \|_A \leqslant \delta_1, \ \| \mu - \mu^g \|_B \leqslant \delta_2,$$
$$\text{if } \delta_1 + \delta_2 = \delta, \text{ then } \| M_T(u_0^{obs}, \mu^g) - u_T^t \|_A \leqslant \varepsilon\}.$$

Following the above idea, we can estimate δ. Investigating the problem

$$\bar{\delta}_{max} = \max_{\delta}\{\delta \mid \| M_T(u_0^{obs}, \mu^g) - M_T(u_0, \mu) \|_A \leqslant \varepsilon,$$
$$u_0 \in B_{\delta_1}, \mu \in B_{\delta_2}, \delta_1 + \delta_2 = \delta\}, \tag{11}$$

we can conclude that

$$\bar{\delta}_{max} \leqslant \delta_{max}.$$

In the abvoe problems, if the errors in the parameter can be ignored, and furthermore the model can be assumed to be perfect, the problems are the three ones of the first kind of predictability; on the other hand, if there exists no initial error, the problems become those of the second kind of predictability concerning the parameter error.

Lorenz model[2] was adopted to demonstrate how to realize these above ideas numerically[16].

4 Nonlinear Optimization Method of Sensitivity Analysis with Numerical Model

The nonlinear optimization method of the sensitivity analysis with numerical model[29] is discussed in detail in this section.

Assume that Eq.(1) is a forecast model with model error, M_t is its propagator, U_0^{obs} and U_T^{obs} are respectively the observational data with error at time 0 and T. How to determine the initial field U_0 that makes the results of the model at time T, $U_T = M_T(U_0)$, can optimally simulate the observational fields, U_T^{obs}. This problem can be formulated into the following nonlinear optimization problem:

$$E = \min_{U_0} J(U_0), \tag{12}$$

where $J(U_0) = \frac{1}{2}(M_T(U_0) - U_T^{obs})^{\mathrm{T}} W (M_T(U_0) - U_T^{obs})$ is the objective function, W is the weighting coefficient matrix.

Let $E = \min J(U_0)$. For a given error bound ε, there are two cases for E,

$$\begin{cases} E > \varepsilon, \\ 0 \leqslant E \leqslant \varepsilon. \end{cases} \tag{13}$$

When testing a model, $E > \varepsilon$ means that even if we get the optimal initial field U_0^*, the model is not able to simulate the observation, U_T^{obs} properly in the given error bound ε. Namely, no matter how we adjust U_0, a satisfactory simulation for U_T^{obs} cannot be obtained. Then we can conclude that the model error is considerably large so that the model needs to be improved. In the case of $0 \leqslant E \leqslant \varepsilon$, the numerical solution $U_T = M_T(U_0)$ and the observation U_T^{obs} have no apparent differences, which indicates that a satisfactory simulation can be obtained by adjusting the initial field U_0. It should be pointed out that, in this case, the model errors could be large too, which will be discussed later. With a given norm, defining a maximum allowable initial error ε_0, we have three cases now:

$$\begin{cases} \| U_0^* - U_0^{obs} \| \ll \varepsilon_0, & (14a) \\ \| U_0^* - U_0^{obs} \| \approx \varepsilon_0, & (14b) \\ \| U_0^* - U_0^{obs} \| \gg \varepsilon_0. & (14c) \end{cases}$$

When the model error is small, the model can simulate the movement of atmosphere very well and we can estimate the observation based on Eqs. (14a), (14b) and (14c). In Eq. (14a), a satisfactory simulation for the observation U_T^{obs} can be obtained from the existing observation U_0^{obs} directly. We do not need to treat the initial field U_0 of the model particularly, and some ordinary interpolation is enough. In Eq. (14b), a satisfactory simulation for U_T^{obs} cannot be obtained from U_0^{obs} directly. But if we improve the initial field U_0 of the model (for example, by assimilation method), a satisfactory simulation can be obtained too. In Eq. (14c), the existing observation lacks enough information and cannot represent the real weather and climate states. If we want to obtain a satisfactory simulation for U_T^{obs}, we should intensify the observational network to get more detailed observation than the existing one.

In case that the observational error is small, the observation is close to the real development of atmosphere. In Eq. (14a), the model error is small and a satisfactory simulation for the observation U_T^{obs} is easily obtained by adjusting the initial field U_0. In Eq. (14b), there are certain model errors but a satisfactory simulation can also be obtained by adjusting U_0 in the allowable error bound. It is the case that an inaccurate model plus an inaccurate initial field produces a satisfactory simulation. In Eq. (14c), the difference between U_0^* and U_0^{obs} is too large, U_0^{obs} is close to the real state, so U_0^* has no physical significance. We can conclude that the model error is large, a satisfactory simulation for U_T^0 is illusive and more work should be done to improve the numerical model.

In practice, it is common to evaluate the model error by comparing the numerical simulation with a relatively accurate observation, or to assess the observation by comparing it with a relatively accurate numerical simulation. In these two cases, we can obtain some significant conclusions by above analysis method. However, due to some objective reasons, when the model and observation are both considerably inaccurate, the applicability of the nonlinear optimization method to sensitivity analysis is limited. In this case, when $E > \varepsilon$, as mentioned above, the model error is considerably large, and the model needs to be improved. When $0 < E \leqslant \varepsilon$, a satisfactory simulation can be obtained by adjusting the initial field U_0, but both model error and observational error may be large. Let ε_0 be the observational precision that is usually known. If $\| U_0^* - U_0^{obs} \| \gg \varepsilon_0$ appears, the optimal initial field U_0^* is far from the real state of atmosphere, and has no physical significance, which implies the model error is large. In other cases, we cannot obtain some significant conclusions by the nonlinear optimization method.

In summary, we can get some instructive conclusions by applying the nonlinear optimization method to sensitivity analysis of a numerical model. Xu *et al*.[30] performed a series of the sensitivity experiments with a two-dimensional quasi-geostrophic model by using the nonlinear optimization methods. The results suggest that the nonlinear optimization method can give a quantitative assessment whether a numerical model is able to simulate the observations and find the initial field that yield the optimal simulation, which are the main advantages of using the nonlinear optimization method. When the simulation results are apparently satisfactory, but the model error and the initial error may be large, under some conditions, the nonlinear optimization method can identify the type of the error that plays a dominant role.

5 Conclusions and Discussions

In this paper, the authors review the applications of nonlinear optimization methods to the predictability study.

Firstly, considering that linear singular vector (LSV) cannot describe the nonlinear motions of atmosphere and ocean, nonlinear singular vector (NSV) and conditional nonlinear optimal perturbation (CNOP) are successively used to investigate the predictability of atmosphere and ocean, and hope that science can gradually understand deeply the nonlinearity on the predictability. The results suggest that NSV and CNOP are useful tools in investigating the effects of nonlinearity on predictability.

Secondly, the new classification of predictability problem provides an approach to estimating quantitatively the maximum predictability time, the maximum prediction error and the allowing maximum initial error and parameter errors of the model. All the information is of importance in the utilization of products of numerical weather and climate prediction.

Thirdly, nonlinear optimization method is applied to the sensitivity analysis with numerical model. It is shown that nonlinear optimization method can give a quantitative assessment whether a numerical model is able to simulate the observations and find the initial field that yields the optimal simulation.

The above results, which are related to the studies of the challenging problems of atmospheric and oceanic sciences, are obtained by using simple model. But it is clear from them that the nonlinear optimization methods can investigate the physical mechanism of atmospheric and oceanic motions, and further explore the nonlinear characteristic of atmospheric and oceanic motions. Therefore it is reasonable to expect that the nonlinear optimization methods will be well applied in complicated models.

In fact a few difficulties will be faced in the utilization of nonlinear optimization methods. For the three sub-problems of predictability from the point of view of pure mathematics, there are still not rather ripe and effective algorithms. In Ref. [16], for the simple models, the lower bounds of maximum predictable time and the maximum allowable initial error are simply computed by filtration. However, the model governing the motions of atmosphere and ocean is generally complicated nonlinear model. For the complicated model, it is very difficult to use this method to solve the corresponding optimization problems. Therefore it is expected that more scientists can enjoy in the development of riper and more effective algorithms to work out the above optimization problems. The estimation of the maximum prediction error, which is equivalent to the calculation of CNOP, for the linear, and the simple quadratic equality or inequality constraint conditions, can respectively be computed by using the ripe limited BFGS algorithm[31] and SQP solver[32]. But, for those nonlinear optimizations problems that are of high dimensions and complex constraint conditions, even of non-smoothed, the above algorithm cannot be used to obtain CNOP effectively. On these problems, there have been some studies of the mathematicians in different aspects. But they are still not well employed in the filed of atmosphere and ocean sciences.

Besides, the applications of the nonlinear optimization method to the sensitivity analysis are also reviewed. This problem belongs to the unconstraint nonlinear optimization problem, which is generally solved by limited BFGS algorithm. However, for the complicated nonlinear optimization problems of high dimensions, the memory (speed, *etc.*) of computer must be considered, which has not been worked out in the previous works.

Therefore it is expected that the rapid development of computer and the combination of computational mathematics and atmospheric science will enable us to achieve our purpose.

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APPLICATIONS OF NONLINEAR OPTIMIZATION METHOD TO NUMERICAL STUDIES OF ATMOSPHERIC AND OCEANIC SCIENCES

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